Tutorial Class 10
Mathematical Methods in Physics

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1. **P9 2014-11-08.** Use the Laplace transform technique and the defining equation of the Green's function to find the Green's function for the equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = f(t) \tag{1}$$

where $y(t) = 0$ for $t < 0$ and $y(0) = 1$, $y'(0) = 3$. Use this Green's function to solve the equation for the case $f(t) = at^2$.

Solution. For the Green's function we need *homogenous* initial conditions so we need to reformulate the ODE in terms of $u(t) = y(t) - v(t)$ where v is one solution to the *homogeneous* equation with the *inhomogeneous* initial conditions. Then we will have

$$u(0) = y(0) - v(0) = 0, \quad u'(0) = y'(0) - v'(0) = 0 \tag{2}$$

since both y and v obey the inhomogeneous initial conditions, so therefore $u(t)$ will have homogeneous boundary conditions, and we can proceed with the standard techniques to find the Green's function $G(t, t')$ remembering that it is the solution for $u(t)$ that can then be written as

$$u(t) = \int G(t, t')f(t')\, dt'. \tag{3}$$

The homogeneous ODE for v is $v'' - 3v' + 2v = 0$ and is solved by an exponential function e^{mt} with m determined by

$$m^2 - 3m + 2 = 0 \quad \Rightarrow \quad m = 1, 2 \tag{4}$$

i.e. $v(t) = Ae^t + Be^{2t}$ with A and B determined by the initial conditions:

$$\begin{cases} 1 = v(0) = A + B \\ 3 = v'(0) = A + 2B \end{cases} \quad \Rightarrow \quad \begin{cases} A = -1 \\ B = 2 \end{cases}. \tag{5}$$

We therefore have

$$v(t) = 2e^{2t} - e^{t}. \tag{6}$$

Therefore we reformulate the ODE in terms of $u(t) = y(t) - v(t) = y(t) + e^t - 2e^{2t}$ and find the corresponding Green's function. Note also that $u(t)$ now indeed obeys homogeneous initial conditions, with $y(0) = 1$, $y'(0) = 3$ we have

$$\begin{aligned} u(0) &= y(0) + 1 - 2 = 0 \\ u'(0) &= y'(0) + 1 - 4 = 0. \end{aligned}$$

We now find the Green's function $G(t, t')$ with homogeneous initial conditions $G(0, t') = 0$, $G'(0, t') = 0$. Since $v(t)$ obeys the homogeneous ODE[1], the ODE:s for $u(t)$ and $y(t)$ have the same right-hand sides $f(t)$, and $u(t)$ is then found by the integral of the product of $G(t, t')$ and the right-hand side of the inhomogeneous ODE for u:

$$u(t) = \int G(t, t') f(t') dt'. \tag{7}$$

The Green's function obeys the differential equation

$$G''(t, t') - 3G'(t, t') + 2G(t, t') = \delta(t - t') \tag{8}$$

Denoting the Laplace transform of G by $\mathcal{L}_L[G(t, t')] = g(s, t')$, we have

$$\mathcal{L}_L[G(t, t')] = g(s, t') = \int_0^\infty G(t, t') e^{-st}\, dt \tag{9}$$

and the transforms of the derivatives are given by

$$\mathcal{L}_L[G'(t, t')] = sg(s, t') - G(0, t') = sg(s, t') \tag{10}$$

$$\mathcal{L}_L[G''(t, t')] = s^2 g(s, t') - sG(0, t') - G'(0, t') = s^2 g(s, t') \tag{11}$$

where we have inserted the homogeneous initial conditions obeyed by $G(t, t')$. The Laplace transform of $\delta(t - t')$ is

$$\mathcal{L}_L[\delta(t - t')] = \int_0^\infty \delta(t - t') e^{-st}\, dt = e^{-st'}, \quad t' \geq 0. \tag{12}$$

We then find that for $t' \geq 0$ the Laplace transformed ODE is

$$s^2 g(s, t') - 3sg(s, t') + 2g(s, t') = e^{-st'} \tag{13}$$

[1]Denoting the differential operator by $\mathcal{L}$, this means that $f(t) = \mathcal{L}[y(t)] = \mathcal{L}[u(t) + v(t)] = \mathcal{L}[u(t)]$ so that u and y have the same right-hand side in the respective ODE:s.

so that the Laplace transformed Green's function is

$$g(s,t') = \frac{e^{-st'}}{s^2 - 3s + 2} = \frac{e^{-st'}}{(s-2)(s-1)} \tag{14}$$

To find the Green's function $G(t,t')$ we must now inverse transform $g(s,t')$. We will here do this using the Bromwich integral, a contour integral along a vertical contour in the complex plane of s. The inverse Laplace transform is then

$$\begin{aligned} G(t,t') &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} g(s,t')\, ds \\ &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{s(t-t')}}{(s-2)(s-1)}\, ds \end{aligned} \tag{15}$$

where the requirement on $\gamma \in \mathbb{R}$ is that $g(s,t')$ is analytic (i.e. has no singularities) for $\mathrm{Re}(s) > \gamma$. Here g has poles at $s = 1$ and $s = 2$ so we must have $\gamma > 2$. To solve the contour integral we will close the contour and use the residue theorem. Depending on the sign of $t - t'$ we will close the contour with a semicircle in the left or right half-plane.

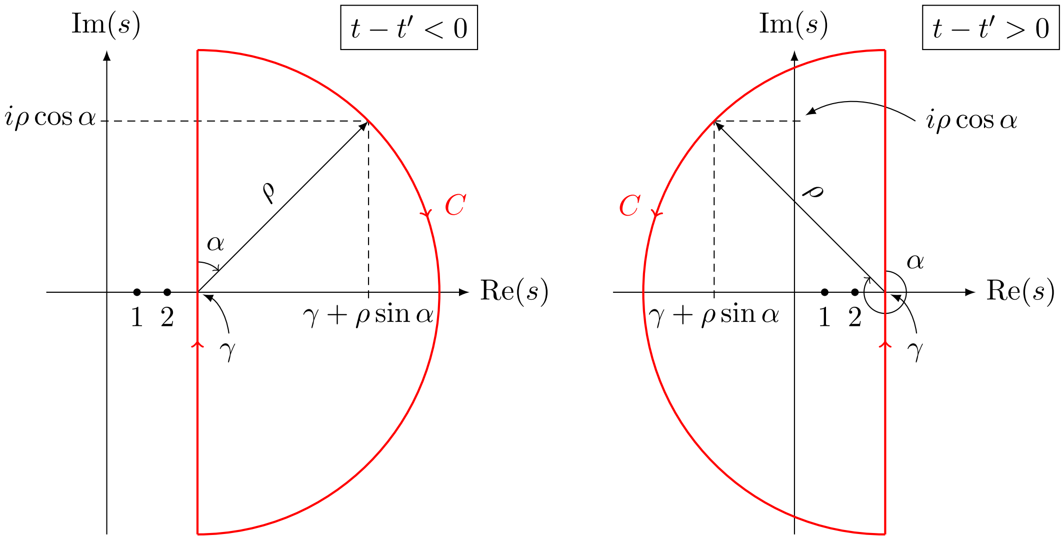

Figure 1: The contours used in P9 2014-11-08. The right one is used for the case $t-t' > 0$ and the left one for $t - t' < 0$.

$\boldsymbol{t - t' < 0.}$ We now close the contour with a large clockwise semicircle in the right half-plane, of radius ρ and centered at $s = \gamma$. Since there are no singularities enclosed (all poles are at points with $\mathrm{Re}(s) < \gamma$) the total contour integral is zero. The contribution from the semicircle will furthermore be zero since the integrand without the exponential goes to zero for large $|s|$ and the exponent in the numerator will always be negative along the circular part. To show this, we denote the integral around the circular part I_ρ where ρ is the radius of the circle and parametrise s on the circle as

$$s = \gamma + \rho(\sin\alpha + i\cos\alpha), \tag{16}$$

where it is important to note that α is here *not* the usual angle measured from the origin, but from the vertical upwards line from the center of the circle at $s = \gamma$. Also, in this parametrisation $|s| \neq \rho$. See Fig. 1 for the contour and the s parametrisation. Using this we find that I_ρ is given by the integral between $0 < \alpha < \pi$. We find, using that on the circle $|ds| = |\rho(\cos\alpha - i\sin\alpha)\, d\alpha| = \rho\, d\alpha$,

$$|I_\rho| \leq e^{\gamma(t-t')} \int_0^\pi \frac{e^{\rho(t-t')\sin\alpha}\rho\; d\alpha}{|\gamma + \rho e^{i\alpha} - 1||\gamma + \rho e^{i\alpha} - 2|}. \tag{17}$$

Now, since $\sin\alpha > 0$ for $0 \leq \alpha \leq \pi$, and we now consider $t - t' < 0$, the exponential goes to zero as the circular radius $\rho \to \infty$ and since the denominator is $\mathcal{O}(\rho^2)$ we have (note that $e^{\gamma(t-t')}$ is just a constant that does not diverge)

$$\lim_{\rho\to\infty} |I_\rho| = 0 \tag{18}$$

and since the total contour integral is zero because there are no poles enclosed, also the Bromwich integral is zero for $t - t' < 0$. We therefore find that

$$G(t, t') = 0, \quad t < t' \tag{19}$$

$\boldsymbol{t - t' > 0}$**.** Now we close instead with a semicircle in the left half-plane (still of radius ρ and centered at $s = \gamma$) since this will make the contribution from the circular part of the contour zero in the same way as above. We now have

$$\begin{aligned} |I_\rho| &\leq e^{\gamma(t-t')} \int_\pi^{2\pi} \frac{e^{\rho(t-t')\sin\alpha}\rho\; d\alpha}{|\gamma + \rho e^{i\alpha} - 1||\gamma + \rho e^{i\alpha} - 2|} \\ &\to 0 \text{ as } \rho \to \infty \end{aligned} \tag{20}$$

since now the fact that instead $\sin\alpha < 0$ in the integration range for α is compensated by the fact that $t - t' > 0$. Then the exponent is negative on the whole semicircle and I_ρ vanishes as $\rho \to \infty$. Again, it is important to note here that the angle α is measured from vertical line going upwards from the centre of the semicircle and *not* from the origin (otherwise $\sin\alpha$ would not in this case be negative in the whole integration).

With the circular part contributing zero to the contour integral, the Bromwich integral is equal to the contour integral around the closed contour and is just $2\pi i$

times the sum of residues of enclosed singularities. The Green's function then becomes

$$G(t,t') = \frac{1}{2\pi i} 2\pi i \sum_i \mathrm{Res}\left(\frac{e^{s(t-t')}}{(s-2)(s-1)}, s_i \right). \tag{21}$$

We have for the simple poles at $s=1$ and $s=2$

$$\mathrm{Res}\left(\frac{e^{s(t-t')}}{(s-2)(s-1)}, s=1 \right) = \lim_{s\to 1}(s-1)\frac{e^{s(t-t')}}{(s-1)(s-2)} = -e^{(t-t')} \tag{22}$$

$$\mathrm{Res}\left(\frac{e^{s(t-t')}}{(s-2)(s-1)}, s=2 \right) = \lim_{s\to 2}(s-2)\frac{e^{s(t-t')}}{(s-1)(s-2)} = e^{2(t-t')} \tag{23}$$

so that the Green's function becomes

$$G(t,t') = \begin{cases} 0, & t < t' \\ e^{2(t-t')} - e^{t-t'}, & t > t' \end{cases} \tag{24}$$

The solution to the inhomogeneous equation for $u(t)$ is now with $f(t) = at^2$

$$\begin{aligned} u(t) &= \int_0^t G(t,t')f(t')\,dt' \\ &= a\left(e^{2t}\int_0^t e^{-2t'}t'^2\,dt - e^t\int_0^t e^{-t'}t'^2\,dt' \right) \end{aligned} \tag{25}$$

Integration by parts twice gives for some r

$$\int_0^t t'^2 e^{-rt'}\,dt' = \ldots = \frac{2}{r^3} - \frac{2}{r^3}e^{-rt} - \frac{2}{r^2}te^{-rt} - \frac{1}{r}t^2e^{-rt} \tag{26}$$

Using this for the two integrals in Eq. (25) results in

$$\begin{aligned} u(t) =& ae^{2t}\left(\frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t} - \frac{1}{2}t^2e^{-2t} \right) \qquad (27) \\ & - ae^t\left(2 - 2e^{-t} - 2te^{-t} - t^2e^{-t}\right) \\ =& \left(-\frac{1}{4}+2\right)a + \left(-\frac{1}{2}+2\right)at + \left(-\frac{1}{2}+1\right)at^2 + \frac{1}{4}ae^{2t} - 2ae^t \\ =& \frac{7}{4}a + \frac{3}{2}at + \frac{1}{2}at^2 + \frac{1}{4}ae^{2t} - 2ae^t \qquad (28) \end{aligned}$$

and therefore

$$\begin{aligned} y(t) &= u(t) + v(t) \\ &= u(t) + 2e^{2t} - e^t \\ &= \frac{7}{4}a + \frac{3}{2}at + \frac{1}{2}at^2 + \left(\frac{1}{4}a + 2\right)e^{2t} - (2a+1)\,e^t \end{aligned} \tag{29}$$

To check we control first that the initial conditions are satisfied. We have

$$y(0) = \frac{7}{4}a + \left(\frac{1}{4}a + 2\right) - (2a + 1) = 1 \tag{30}$$

$$y'(0) = \frac{3}{2}a + 2\left(\frac{1}{4}a + 2\right) - (2a + 1) = 3 \tag{31}$$

and they are satisfied. Checking that $y(t)$ satisfies the ODE we have

$$\begin{aligned}\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = a &+ ae^{2t} - 2ae^t \\ &- \frac{9}{2}a - 3at - \frac{3}{2}ae^{2t} - 6ae^t \\ &+ \frac{7}{2}a + 3at + at^2 + \frac{1}{2}ae^{2t} - 4ae^t \\ = at^2& \\ = f(t)& \end{aligned} \tag{32}$$

so that $y(t)$ indeed is our solution.

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2. **P11.9.6.**

   (a) Using a method based on contour integration, evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

   (b) Check your work by relating your answer to an appropriate expression involving zeta functions.

Solution.

(a) We are going to evaluate the sum using residue calculus. The idea is to consider the sum as a sum of residues and, following the residue theorem, set this equal to a contour integral that encloses all the poles. We then have to be a bit clever and find a function that has an infinite number of poles (because the sum goes to infinite n) with residues corresponding to the expression in the sum.

To do this, we note that the function $\pi \tan \pi z$ has simple poles along the real line at $z = n + \frac{1}{2}$ where n is an integer, and the residue of each pole is -1 since

$$\lim_{z \to n + \frac{1}{2}} \left(z - n - \frac{1}{2}\right) \pi \tan \pi z = -1 \tag{33}$$

which can be seen by expanding the trigonometric functions around $z = n + \frac{1}{2}$. This means that if we integrate $f(z)\pi \tan \pi z$ for some function $f(z)$ around a contour which is a large circle of radius $N+\frac{1}{2}$ (centered around $z=\frac{1}{2}$ so that it encloses the poles of $\pi\tan\pi z$ between $z=-N+\frac{1}{2}$ and $z=N+\frac{1}{2}$) we will have by the residue theorem, denoting the contour integral I,

$$\begin{aligned} I =& 2\pi i \sum_{n=-N}^{N}(-1)f(n+\frac{1}{2}) \\ &+ 2\pi i \sum_j (\text{ residues of } f(z)\pi\tan\pi z \text{ at singularities } z_j \text{ of } f(z)) \end{aligned} \tag{34}$$

where the residue of the integrand $f(z)\pi\tan\pi z$ is $f(n+\frac{1}{2})$ at $z=n+\frac{1}{2}$ for integer n and the second line comes from the fact that the contour may apart from the poles of $\pi\tan\pi z$ also enclose points where $f(z)$ is singular. The integral around this circle will go to zero as $N\to\infty$ if $f(z)$ is such that $zf(z)\to 0$ for $|z|\to\infty$. Therefore, when that condition is met, $I=0$ and

$$\sum_{n=-\infty}^{\infty} f(n+\frac{1}{2}) = \sum_j (\text{ residues of } f(z)\pi\tan\pi z \text{ at singularities } z_j \text{ of } f(z))\,. \tag{35}$$

In this case we see that we want to choose

$$f(z)=\frac{1}{z^2} \tag{36}$$

in order to get the sum on the left-hand side of Eq. (35). The sum we want to evaluate is however not quite that sum, since n should start from $n=0$ and not $n=-\infty$. Writing the sum we want to evaluate as

$$S=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^2} \tag{37}$$

we note that

$$\begin{aligned} \sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2} &= \sum_{n=-\infty}^{-1}\frac{1}{(2n+1)^2}+S \\ &= \sum_{n=-\infty}^{0}\frac{1}{(2(n-1)+1)^2}+S \\ &= \sum_{n=-\infty}^{0}\frac{1}{(2n-1)^2}+S \\ &= \sum_{n=0}^{\infty}\frac{1}{(2(-n)-1)^2}+S \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}+S \\ &= 2S \end{aligned} \tag{38}$$

so that the sum on the left-hand side of Eq. (35) is related to S as

$$S = \frac{1}{2}\frac{1}{4}\sum_{n=-\infty}^{\infty}\frac{1}{(n+\frac{1}{2})^2} = \frac{1}{8}\sum_{n=-\infty}^{\infty}\frac{1}{(n+\frac{1}{2})^2} \tag{39}$$

and with $f(z) = 1/z^2$ we need to evaluate the residue of $f(z)\pi\tan\pi z$ at the double pole of $f(z)$ at $z = 0$ to evaluate S. The residue there is given by

$$\begin{aligned}
\operatorname{Res}(f(z), z = 0) &= \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}\left[z^2\frac{\pi\tan\pi z}{z^2}\right] \\
&= \lim_{z\to 0}\frac{\pi^2}{\cos^2\pi z} \\
&= \pi^2
\end{aligned} \tag{40}$$

and the sum becomes

$$S = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \tag{41}$$

(b) The zeta function is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \quad \operatorname{Re}(z) > 1. \tag{42}$$

To evaluate our sum S in terms of values of $\zeta(z)$ we must rewrite S in terms of zeta functions. As a start, we have

$$\begin{aligned}
S &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\
&= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \\
&= \sum_{k=1,3,5,\ldots}\frac{1}{k^2}
\end{aligned} \tag{43}$$

where we have in the second line shifted index by one step and in the last line changed from n to the summation variable $k = 2n - 1$, i.e. $n = (k+1)/2$, so that the sum goes over odd k given by $k = 1, 3, 5, \ldots$. The sum in the last line is similar to $\zeta(2)$, but with the difference that the even k are not included in the sum. We therefore add and subtract $\sum_{k=2,4,6,\ldots}(1/k^2)$ to find

$$\begin{aligned}
S &= \left[\sum_{k=1,3,5,\ldots} \frac{1}{k^2} + \sum_{k=2,4,6,\ldots} \frac{1}{k^2}\right] - \sum_{k=2,4,6,\ldots} \frac{1}{k^2} \\
&= \zeta(2) - \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \ldots\right) \\
&= \zeta(2) - \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \ldots\right) \\
&= \zeta(2) - \frac{1}{4}\zeta(2) \\
&= \frac{3}{4}\zeta(2) \qquad (44)
\end{aligned}$$

where we have identified a second occurence of $\zeta(2)$ inside the brackets in the third line. Given that $\zeta(2) = \pi^2/6$ we then find

$$S = \frac{3}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8} \qquad (45)$$

which agrees with what we found in (a) above.

3. **P11.5.6-11.5.8** Obtain the Laurent expansion of $f(z)$ around z_0 where

11.5.6. $f(z) = \frac{e^z}{z^2}$, $z_0 = 0$

11.5.7. $f(z) = \frac{ze^z}{z-1}$, $z_0 = 1$

11.5.8. $f(z) = (z-1)e^{1/z}$, $z_0 = 0$

Solution.

11.5.6. Since e^z is an entire function (analytic everywhere) we expect there to be a double pole at $z = 0$ from the denominator. Using the Taylor expansion of e^z around $z = 0$ we have

$$\begin{aligned}
\frac{e^z}{z^2} &= \frac{1}{z^2}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right) \\
&= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{3!} + \ldots \\
&= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+2)!} \qquad (46)
\end{aligned}$$

and since the last sum does not contain any poles at $z = 0$, we see that there is indeed a double pole at $z = 0$ since the Laurent expansion for negative powers of z terminates at $1/z^2$.

11.5.7. Now we should do the expansion around $z = 1$. It is not as easy as in the previous problem to see what type of singularity $f(z)$ has at $z = 1$ in this case, but given that the numerator is well-behaved at $z = 1$, we expect a simple pole at $z = 1$ from the denominator. To find the Laurent series we can write $e^z = e^{z-1}e$ and $z = (z-1)+1$, this results in

$$\begin{aligned} f(z) &= \frac{ze^z}{z-1} \\ &= \frac{((z-1)+1)ee^{z-1}}{z-1} \\ &= e\left(1+\frac{1}{z-1}\right)e^{z-1} \end{aligned} \tag{47}$$

We now use the expansion of e^{z-1} around $z = 1$, then

$$\begin{aligned} f(z) &= e\left(1+\frac{1}{z-1}\right)\sum_{n=0}^{\infty}\frac{(z-1)^n}{(n)!} \\ &= e\left[\sum_{n=0}^{\infty}\frac{(z-1)^n}{(n)!}+\sum_{n=0}^{\infty}\frac{(z-1)^{n-1}}{(n)!}\right] \\ &= e\left[\sum_{n=0}^{\infty}\frac{(z-1)^n}{(n)!}+\sum_{n=-1}^{\infty}\frac{(z-1)^n}{(n+1)!}\right] \\ &= e\left[\frac{1}{z-1}+\sum_{n=0}^{\infty}\left(\frac{1}{n!}+\frac{1}{(n+1)!}\right)(z-1)^n\right] \\ &= \frac{e}{z-1}+e\sum_{n=0}^{\infty}\left(\frac{n+2}{n+1}\right)\frac{(z-1)^n}{n!} \end{aligned} \tag{48}$$

where we shifted the summation index in the third equality to combine the sums. We note that the singularity at $z = 1$ is indeed a simple pole as anticipated.

11.5.8. $e^{1/z}$ has an essential singularity at $z = 0$, meaning that an infinite number of negative powers of z appear in the Laurent expansion. To see this we expand $e^{1/z}$ in powers of $1/z$:

$$\begin{aligned} e^{1/z} &= \sum_{n=0}^{\infty}\frac{z^{-n}}{n!} \\ &= 1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3!z^3}+\dots \end{aligned} \tag{49}$$

To find the Laurent expansion of $(z-1)e^{1/z}$ around $z = 0$ we multiply the above expansion by $z-1$ and get

$$
\begin{aligned}
(z-1)e^{1/z} &= \sum_{n=0}^{\infty} \frac{z^{-n+1}}{n!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \\
&= \sum_{n=-1}^{\infty} \frac{z^{-n}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \\
&= z + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - 1 \right) \frac{z^{-n}}{n!} \\
&= z + \sum_{n=1}^{\infty} \frac{n}{n+1} \frac{z^{-n}}{n!} \qquad (50)
\end{aligned}
$$

where we have shifted the summation index in the second line, written out the $n=-1$ term explicitly and combined the two sums in the third line and used that the $n=0$ term is zero in the last line so that the sum can start from $n=1$. We note that the sum includes infinitely negative powers of z, which means that we have expanded around an essential singularity (i.e. an "infinite order" pole, a term which is however not used).

4. **P7 2016-01-20.**

   (a) Evaluate the ratio $\Gamma\left(\frac{2}{5}\right)/\Gamma\left(\frac{12}{5}\right)$ where $\Gamma(p)$ is the Gamma function.

   (b) Evaluate in terms of the Gamma function the integral $\int_0^1 \sqrt[3]{\ln x}\, dx$.

   Solution.

   (a) A fundamental property of the Gamma function is $\Gamma(z+1) = z\Gamma(z)$. Using this here we have

   $$\Gamma\left(\frac{12}{5}\right) = \frac{7}{5}\Gamma\left(\frac{7}{5}\right) = \frac{7}{5}\frac{2}{5}\Gamma\left(\frac{2}{5}\right) = \frac{14}{25}\Gamma\left(\frac{2}{5}\right). \qquad (51)$$

   The ratio is then

   $$\frac{\Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{12}{5}\right)} = \frac{\Gamma\left(\frac{2}{5}\right)}{\frac{14}{25}\Gamma\left(\frac{2}{5}\right)} = \frac{25}{14} \qquad (52)$$

   (b) The Euler definite integral definition of $\Gamma(z)$ is

   $$\Gamma(z) \equiv \int_0^{\infty} e^{-t} t^{z-1}\, dt, \quad \mathrm{Re}(z) > 0. \qquad (53)$$

   Making the variable substitution $\ln x = -t$ we have $x = e^{-t}$ and $dx = -e^{-t}dt$. When $x \to 0$, $t \to \infty$ and when $x = 1$, $t = 0$ so

   $$\int_0^1 \sqrt[3]{\ln x}\, dx = -\int_{\infty}^{0} \sqrt[3]{-t} e^{-t}\, dt = -\int_0^{\infty} e^{-t} t^{1/3}\, dt = -\Gamma\left(\frac{4}{3}\right) \qquad (54)$$

   where $\sqrt[3]{-t} = -t^{1/3}$ for t real is the (unique) cubic root of $-t$.

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5. **P3, exam 2011-01-05.** [COPIED FROM TC2.]

   (a) Use the Frobenius method to find one solution to the differential equation (note: it is a simple closed form): $x(x+1)\,y'' - (x-1)\,y' + y = 0$.

   (b) The Fuchsian conditions state that for the general second-order differential equation $y'' + f(x)y' + g(x)y = 0$ to give two independent solutions with the method of Frobenius it is necessary and sufficient that $xf(x)$ and $x^2g(x)$ are expandable in convergent power series $\sum_{n=0}^{\infty} a_n x^n$. Show that the Fuchsian conditions hold for the equation in (a) above.

   (c) When the Fuchsian conditions hold,the general solution by the method of Frobenius will be either (i) two Frobenius series or (ii) one solution which is a Frobenius series $S_1(x)$ and a second solution $S_1(x)\ln x + S_2(x)$ where $S_2(x)$ is a second Frobenius series. Use case (ii) to find a second solution to the equation in (a) above.

   Solution.

   (a) In the Frobenius method we assume a series solution of form $y = \sum_{j=0}^{\infty} a_j x^{s+j}$, $a_0 \neq 0$, and insert this into the ODE and work out the solution from there. From Fuchs' theorem we are guaranteed that this will give as at least one of the two solutions to the second order ODE as long as we are expanding about a point that is at worst a regular singularity. With the series ansatz we find

$$
\begin{aligned}
x(x+1)y'' &= \sum_{j=0}^{\infty} \Big[a_j(s+j)(s+j-1)x^{s+j} + a_j(s+j)(s+j-1)x^{s+j-1}\Big] \\
&= a_0 s(s-1)x^{s-1} + \sum_{j=1}^{\infty} \Big[a_{j-1}(s+j-1)(s+j-2) \\
&\qquad + a_j(s+j)(s+j-1)\Big]x^{s+j-1} \qquad (55) \\
-(x-1)y' &= \sum_{j=0}^{\infty} \left(-a_j(s+j)\right) x^{s+j} + \sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} \\
&= a_0 s x^{s-1} + \sum_{j=1}^{\infty} \Big[-a_{j-1}(s+j-1) + a_j(s+j)\Big]x^{s+j-1} \qquad (56) \\
y &= \sum_{j=0}^{\infty} a_j x^{s+j} = \sum_{j=1} a_{j-1}x^{s+j-1} \qquad (57)
\end{aligned}
$$

Inserting this into the ODE yields

$$a_0(s(s-1)+s)x^{s-1} + \sum_{j=1}^{\infty} \Big[(s+j-1)\left((s+j-2)a_{j-1} + (s+j)a_j\right) - a_{j-1}(s+j-1) + a_j(s+j) + a_{j-1}\Big] x^{s+j-1} = 0 \quad (58)$$

and the coefficients must vanish for each power of x separately. The lowest power of x is x^{s-1}, requiring this coefficient to be zero results in the indicial equation:

$$a_0(s(s-1)+s) = a_0 s^2 = 0 \quad \Leftrightarrow \quad s = 0 \quad (59)$$

We have a double root to the indicial equation, so we can only get one series solution in this way. We will look at the other solution, which contains a logarithmic term, later. For $j \geq 1$ we get a recursion relation:

$$\begin{aligned} &(s+j-1)\left((s+j-2)a_{j-1} + (s+j)a_j\right) - a_{j-1}(s+j-1) + a_j(s+j) + a_{j-1} = 0 \\ \Leftrightarrow \quad & a_j = -a_{j-1} \frac{(j+s-2)^2}{(j+s)^2} \end{aligned} \quad (60)$$

which with $s = 0$ becomes

$$a_j = -a_{j-1} \frac{(j-2)^2}{j^2}, \quad j = 1, 2, \ldots \quad (61)$$

The series coefficients become

$$j = 1: \quad a_1 = -a_0 \quad (62)$$

$$j = 2: \quad a_2 = -a_1 \cdot 0 \quad \Rightarrow \quad a_2 = a_3 = a_4 = \ldots = 0 \quad (63)$$

so the first solution becomes rather simple, it is

$$y_1(x) = \sum_{j=0}^{\infty} a_j x^j = a_0 + a_1\, x = a_0\, (1-x) \quad (64)$$

(b) The Fuchsian condition guarantees two solutions to our ODE if $xf(x)$ and $x^2 g(x)$ are analytic around the point which we are expanding, i.e. expandable as Taylor series, where the ODE is written as

$$y'' + f(x)y' + g(x)y = 0 \quad (65)$$

so that we have in our case

$$f(x) = \frac{1-x}{x(1+x)}, \quad g(x) = \frac{1}{x(1+x)}. \quad (66)$$

We will now check that the Fuchsian condition holds. Using the binomial series expansion[2] we get

$$xf(x) = \frac{1-x}{1+x} = (1-x)\sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1 \tag{67}$$

$$x^2 g(x) = \frac{x}{1+x} = x\sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1 \tag{68}$$

So the Fuchsian condition holds for $|x| < 1$, when the series converges.

(c) We now consider the second solution. The second solution is of the form

$$y_2(x) = y_1(x)\ln x + \sum_{j=0} b_j x^j. \tag{69}$$

To determine the behaviour of the b_j we insert y_2 into the ODE and simplify, this will give recursive relations between the b_j and the a_j. The derivatives become

$$y_2' = y_1' \ln x + \frac{y_1}{x} + \sum_{j=0}^{\infty} b_j j x^{j-1} \tag{70}$$

$$y_2'' = y_1'' \ln x + \frac{y_1'}{x} + \frac{y_1'}{x} - \frac{y_1}{x^2} + \sum_{j=0}^{\infty} b_j j(j-1)x^{j-2} \tag{71}$$

Inserting this into the ODE we find

$$\begin{aligned}
&\underbrace{\mathcal{L}[y_1]}_{=0}\ln x + 2(x+1)y_1' \underbrace{-\frac{x+1}{x}y_1 - \frac{x-1}{x}y_1}_{=-2y_1} + \mathcal{L}\left[\sum_j b_j x^j\right] = 0 \\
\Rightarrow \quad & 2(x+1)y_1' - 2y_1 + \mathcal{L}\left[\sum_j b_j x^j\right] = 0
\end{aligned} \tag{72}$$

where the first term in the first expression is zero since y_1 is a solution to the

[2] The binomial expansion, convergent for $|x| < 1$, says that $(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$ and for $\alpha = -1$ we have

$$\binom{-1}{n} = \frac{(-1)(-2)\cdots(-1-n+1)}{n!} = \frac{(-1)^n n!}{n!} = (-1)^n.$$

ODE so that $\mathcal{L}[y_1] = 0$. Operating with $\mathcal{L}$ on the series gives

$$\begin{aligned}
\mathcal{L}\left[\sum_j b_j x^j\right] &= \sum_j b_j j(j-1)x^j + \sum_j b_j j(j-1)x^{j-1} \\
&\quad - \sum_j b_j j x^j + \sum_j b_j j x^{j-1} + \sum_j b_j x^j \\
&= \sum_j b_j j^2 x^{j-1} + \sum_j b_j (j-1)^2 x^j
\end{aligned} \tag{73}$$

where all sums go from $j = 0$. Using the form for y_1 and $y_1' = -a_0$ we finally get from Eq. (72) the equation

$$4a_0 = \sum_{j=0}^{\infty} b_j j^2 x^{j-1} + \sum_{j=0}^{\infty} b_j (j-1)^2 x^j \tag{74}$$

This equation must hold power for power in x. We start with the lowest power of x and continue upwards. We find[3]

$$x^{-1}: \quad 0 = b_0 \cdot 0 \quad \Rightarrow \quad b_0 \text{ undetermined} \tag{75}$$

$$x^0: \quad 4a_0 = b_1 \cdot 1^2 + b_0(0-1)^2 \quad \Rightarrow \quad b_1 = 4a_0 - b_0 \tag{76}$$

For x^j with $j \geq 1$ we get a recursion relation for the b_j:

$$x^j,\ j \geq 1: \quad 0 = b_{j+1}(j+1)^2 + b_j(j-1)^2 \quad \Rightarrow \quad b_{j+1} = -b_j \left(\frac{j-1}{j+1}\right)^2 \tag{77}$$

and we see that the series terminates after $j = 1$ since

$$b_2 = -b_1 \cdot 0 \quad \Rightarrow \quad b_2 = b_3 = b_4 = \ldots = 0 \tag{78}$$

and therefore the second solution is given by

$$y_2(x) = y_1(x)\ln x + \sum_{j=0}^{\infty} b_j x^j = a_0(1-x)\ln x + b_0 + (4a_0 - b_0)x \tag{79}$$

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[3] We have no parity arguments to put $b_0 = 0$ in this case, $\mathcal{L}$ has no definite parity.