Tutorial Class 1
Mathematical Methods in Physics

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1. **P7, exam 2012-11-10.** Solve the following ODE:

$$(1+x^2)\frac{dy}{dx} + 6xy = 2x \tag{1}$$

Solution. The ODE is *separable*, so we can separate variables and integrate to obtain an implicit solution for y, that we can then solve to get the explicit solution (the last step is not always easy or possible).

The ODE is separable if we can write it as

$$\frac{dy}{dx} = -\frac{P(x)}{Q(y)} \quad \Leftrightarrow \quad P(x)\,dx + Q(y)\,dy = 0 \tag{2}$$

for some functions P and Q. Here we have:

$$P(x) = \frac{2x}{1+x^2},\ Q(y) = \frac{1}{3y-1} \tag{3}$$

i.e.

$$\frac{dy}{dx} = -\frac{2x/(1+x^2)}{1/(3y-1)} \tag{4}$$

and we see that the ODE is indeed separable. We then solve the ODE by integrating from (x_0, y_0) to (x, y) and then obtain

$$\int_{y_0}^{y} \frac{dy}{3y-1} = -\int_{x_0}^{x} \frac{2x\,dx}{1+x^2}. \tag{5}$$

The lower limits integrate to constants that we can combine into a single constant C. We get

$$\frac{1}{3}\ln(3y-1) = -\ln(1+x^2) + C \tag{6}$$

which can be simplified into

$$3y - 1 = C\frac{1}{(1+x^2)^3} \tag{7}$$

where C has now changed value and

$$y = \frac{1}{3} + \frac{C}{3(1+x^2)^3} \tag{8}$$

where the value of C is again different.

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2. **P7.3.3, Arfken, Weber & Harris.** Find the general solution to

$$y''' - 3y' + 2y = 0. \tag{9}$$

Write the solution in terms of real quantities only.

Solution. This is a third order homogeneous ODE with constant coefficients. The solutions will then be of the form $y_i = e^{m_i x}$ where m_i are the roots of the algebraic equation obtained by inserting e^{mx} into the ODE. With

$$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}, \quad y''' = m^3 e^{mx} \tag{10}$$

inserted into the ODE we get the algebraic equation

$$m^3 - 3m + 2 = 0. \tag{11}$$

We can immediately see that $m = 1$ is a root. Factorising the equation (using e.g. polynomial long division) we find that

$$m^3 - 3m + 2 = (m-1)(m^2 + m - 2) = 0 \tag{12}$$

and the second order equation $m^2 + m - 2 = 0$ has roots $m = -2$ and $m = 1$, i.e. $m = 1$ is a double root. Since the ODE is third order it has three linearly independent solutions, and in the case of a double root $m_1 = 1$ the two solutions will be

$$y_1 = e^{m_1 x} = e^x, \quad y_2 = \frac{dy_1}{dm_1} = xe^{m_1 x} = xe^x \tag{13}$$

where we have inserted $m = 1$ in the last equalities. In general, for an n:th order root, the n solutions are given by

$$y_\ell = \frac{d^\ell}{dm^\ell} e^{mx}, \quad \ell = 1, \ldots, n \tag{14}$$

The third solution belonging to the root $m = -2$ is

$$y_3 = e^{-2x} \tag{15}$$

and the general solution to the ODE is given by the sum of the three y_i with arbitrary coefficients,

$$y = C_1 e^x + C_2 x e^x + C_3 e^{-2x} \tag{16}$$

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3. **P3, exam 2013-11-09.** Consider the following differential equation:

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (E-1)y = 0 \tag{17}$$

derived from the quantum mechanical harmonic oscillator.

(a) Use Frobenius method to find the odd and even solutions for this equation.

(b) Determine the values of the energy E for which the series terminate, resulting in polynomials of finite order.

(c) Write down explicitly the polynomials corresponding to the three lowest energies as obtained from your expansion and give their energies. (The units are arbitrary here.)

Solution.

(a) In the Frobenius method we find the solution by assuming a power series solution around a point x_0, usually taken to be $x_0 = 0$. In this case $x_0 = 0$ is an ordinary point so expanding around 0 gives at least one solution (see Fuchs' theorem). We thus try the solution

$$y(x) = x^s(a_0 + a_1 x + a_2 x^2 + \ldots) = \sum_{j=0}^{\infty} a_j x^{s+j} \tag{18}$$

where $a_0 \neq 0$, i.e. the first non-zero term goes as x^s, where s need not be an integer. We differentiate and obtain

$$\frac{dy}{dx} = \sum_j a_j(s+j)x^{s+j-1}, \quad \frac{d^2y}{dx^2} = \sum_j a_j(s+j)(s+j-1)x^{s+j-2}. \tag{19}$$

Substituting this into the ODE results in

$$\sum_j a_j(s+j)(s+j-1)x^{s+j-2} - 2\sum_j a_j(s+j)x^{s+j} + (E-1)\sum_j a_j x^{s+j} = 0 \tag{20}$$

The coefficient of each power of x on the left-hand side must therefore vanish individually, so we obtain a set of equations, one for each value of j that must be satisfied. The lowest power of x occurring is x^{s-2} in the first sum with $j = 0$ (this power of x does not show up in the last two sums). Requiring that the coefficient of x^{s-2} is zero results in the equation

$$a_0 s(s-1) = 0 \tag{21}$$

and since we have chosen a_0 to be the coefficient of the lowest non-zero term we have $a_0 \neq 0$ and the *indicial equation* becomes

$$s(s-1) = 0 \tag{22}$$

with solutions $s = 0$ or $s = 1$. From Fuchs' theorem we are guaranteed that there is a series solution for the larger root $s = 1$ but since the difference between the roots is an integer we are not guaranteed that $s = 0$ will give a series solution. For the coefficient of the next power of x (i.e. x^{s+j-1}) to vanish we get another similar equation requiring (again only the first sum contributes) that

$$a_1(s+1)s = 0 \tag{23}$$

i.e. that we must set $a_1 = 0$ for $s = 1$ and and can set $a_1 = 0$ for $s = 0$.
For the remaining coefficients to vanish we must have (the first sum contributes with $j+2$ terms to the power of x^{j+s})

$$a_{j+2}(s+j+2)(s+j+1) - 2a_j(s+j) + (E-1)a_j = 0 \tag{24}$$

Rearranging this we get a *recursion relation* for the a_j:

$$a_{j+2} = a_j \frac{2(s+j) + 1 - E}{(j+s+2)(j+s+1)} \tag{25}$$

We see now that for $a_1 = 0$ all odd terms $a_1 = a_3 = a_5 = \ldots = 0$ and only even j contribute. We then get two solutions, y_{even} only containing even powers of x for $s = 0$ and y_{odd} only containing odd powers for $s = 1$.[1] This is also a consequence of the fact that the differential operator in the ODE,

$$\mathcal{L}(x)y(x) = \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (E-1)y \tag{26}$$

has definite parity, i.e. $\mathcal{L}(-x)y(-x) = +\mathcal{L}(x)y(-x)$ and therefore the solutions of the ODE can be written as one even and one odd function of x, linearly independent of each other. With $a_1 = 0$, the solution for $s = 1$, y_{odd},

[1] With $a_1 \neq 0$ for $s = 0$ we would get odd powers in the solution, but these are just a multiple of the odd solution and we can then remove this and put into the odd solution instead.

will contain only odd powers of x and the solution with $s = 0$, y_{even}, contains only even powers.

For $s = 1$ we get for $y_{\text{odd}}(x) = \sum_{j \text{ even}} a_j x^{1+j}$

$$\begin{aligned}
a_2 &= a_0 \frac{(3-E)}{6} = a_0 \frac{(3-E)}{3!} \\
a_4 &= a_0 \frac{(3-E)}{6} \frac{(7-E)}{20} = a_0 \frac{(3-E)(7-E)}{5!} \\
a_6 &= a_0 \frac{(3-E)(7-E)}{120} \frac{(11-E)}{42} = a_0 \frac{(3-E)(7-E)(11-E)}{7!} \\
&\vdots
\end{aligned} \tag{27}$$

whereas for $s = 0$ we have (expanding the other solution as $y_{\text{even}}(x) = \sum_{j \text{ even}} b_j x^j$)

$$\begin{aligned}
b_2 &= b_0 \frac{(1-E)}{2} = b_0 \frac{(1-E)}{2!} \\
b_4 &= b_0 \frac{(1-E)}{2} \frac{(5-E)}{12} = b_0 \frac{(1-E)(5-E)}{4!} \\
b_6 &= b_0 \frac{(1-E)(5-E)}{24} \frac{(9-E)}{30} = b_0 \frac{(1-E)(5-E)(9-E)}{6!} \\
&\vdots
\end{aligned} \tag{28}$$

and the $s = 0$ case gave a series solution in this case since the b_j are well-behaved[2].

The solutions are

$$\begin{aligned}
y_{\text{odd}}(x) = a_0 \Big(x &+ \frac{3-E}{3!} x^3 + \frac{(3-E)(7-E)}{5!} x^5 \\
&+ \frac{(3-E)(7-E)(11-E)}{7!} x^7 + \dots \Big)
\end{aligned} \tag{29}$$

$$\begin{aligned}
y_{\text{even}}(x) = b_0 \Big(1 &+ \frac{1-E}{2!} x^2 + \frac{(1-E)(5-E)}{4!} x^4 \\
&+ \frac{(1-E)(5-E)(9-E)}{6!} x^6 + \dots \Big)
\end{aligned} \tag{30}$$

(b) From the recursion relation we find that the series terminates if

$$\frac{a_{j+2}}{a_j} = \frac{2(s+j)+1-E}{(s+j+2)(s+j+1)} = 0 \tag{31}$$

for some s, j and E, i.e. if

$$E = 1 + 2(s+j). \tag{32}$$

[2] In other cases with other ODE:s the b_j can for example diverge at some j making a series solution invalid.

With $s = 0, 1$ and $j = 0, 2, 4, \ldots$ this means that E has to satisfy

$$s = 0: \quad E = 1, 5, 9, \ldots \tag{33}$$
$$s = 1: \quad E = 3, 7, 11, \ldots \tag{34}$$

(c) The three lowest energies are $E_0 = 1$, $E_1 = 3$ and $E_2 = 5$, with the first and third belonging to the even solution with $s = 0$ and the second to the odd solution with $s = 1$. The polynomials are

$$E_0 = 1: \quad y_0(x) = b_0\, 1 \tag{35}$$
$$E_1 = 3: \quad y_1(x) = a_0\, x \tag{36}$$
$$E_2 = 5: \quad y_2(x) = b_0\,(1 - 2x^2) \tag{37}$$

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4. **P7.4.1, Arfken, Weber & Harris.** Show that Legendre's differential equation $(1 - x^2)y'' - 2xy' + l(l+1)y = 0$ has regular singularities at -1, 1, and ∞.

Solution. A point x_0 is a *regular singular point* if the functions $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are both analytic at x_0, which is the same as saying that the limits

$$\lim_{x \to x_0} (x - x_0)p(x), \quad \lim_{x \to x_0} (x - x_0)^2 q(x) \tag{38}$$

exist and are finite where the ODE is formulated in standard form as

$$y'' + p(x)y' + q(x)y = 0. \tag{39}$$

Here we find that

$$p(x) = \frac{-2x}{1 - x^2}, \quad q(x) = \frac{l(l+1)}{1 - x^2} \tag{40}$$

- $x = -1$

$$\lim_{x \to -1} (x+1)\frac{-2x}{1 - x^2} = \lim_{x \to -1} (x+1)\frac{-2x}{(1-x)(1+x)} = \lim_{x \to -1} \frac{2x}{x - 1} = 1 \tag{41}$$
$$\lim_{x \to -1} (x+1)^2\frac{l(l+1)}{1 - x^2} = \lim_{x \to -1} (x+1)\frac{l(l+1)}{1 - x} = 0 \tag{42}$$

so both limits exist and are finite and $x = -1$ is a regular singularity.

- $x = 1$

$$\lim_{x \to 1} (x-1)\frac{-2x}{1 - x^2} = \lim_{x \to 1} \frac{+2x}{1 + x} = 1 \tag{43}$$
$$\lim_{x \to 1} (x-1)^2\frac{l(l+1)}{1 - x^2} = \lim_{x \to 1} (1-x)\frac{l(l+1)}{1 + x} = 0 \tag{44}$$

which shows that $x = 1$ is a regular singularity.

- $x \to \infty$
  To analyse this point we express the ODE in terms of $z = 1/x$ and look at the behaviour as $z \to 0$. We must then express the derivatives in terms of the new variable properly and when this is done we get new expressions instead of just $p(x)$ and $q(x)$ in the limits. The limits that must exist and be finite for $x \to \infty$ to be a regular singularity are then (see Arfken, Weber & Harris p. 344 for a full discussion):

$$\lim_{z\to 0}(z-0)\,\frac{2z-p(1/z)}{z^2}, \quad \lim_{z\to 0}(z-0)^2\,\frac{q(1/z)}{z^4} \tag{45}$$

  We find

$$\lim_{z\to 0} z\,\frac{2z-(-2z/(z^2-1)}{z^2} = \lim_{z\to 0} 2(1+\frac{1}{z^2-1}) = 0 \tag{46}$$

$$\lim_{z\to 0} z^2\,\frac{l(l+1)z^2/(z^2-1)}{z^4} = \lim_{z\to 0}\frac{l(l+1)}{z^2-1} = -l(l+1) \tag{47}$$

  and since both limits exist and are finite, $x \to \infty$ is a regular singularity of the ODE (as expressed originally in terms of x).