# Tutorial Class 8
# Mathematical Methods in Physics

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*Last typeset:* October 15, 2018

## Spherical harmonics and the Schrödinger equation

1. The time-independent Schrödinger equation for a spherically symmetric potential $V(r)$ can be written for a wave function $\psi(r, \theta, \varphi)$ as

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi \tag{1}$$

where $E$ is the energy and $m$ the particle mass.

   (a) Separate the radial and angular part assuming $\psi = R(r)Y(\theta, \varphi)$ and set the separation constant to $\ell(\ell + 1)$ to find the differential equations obeyed by $R(r)$ and $Y(\theta, \varphi)$.

   (b) Separate variables again, assuming $Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi)$ with separation constant $m^2$ to obtain the ordinary differential equations for $\Theta$ and $\Phi$. What are the solutions for $\Theta$ and $\Phi$?

   (c) How are the spherical harmonics $Y_\ell^m$ defined?

   (d) What is the partial differential equation that the spherical harmonics $Y_\ell^m$ obey? (There will be one for each $m$ and $\ell$.)

2. The angular momentum operator $\mathbf{L} = (L_x, L_y, L_z)$ in quantum mechanics is in spherical coordinates given by

$$\mathbf{L} = -i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi},\ \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi},\ \frac{\partial}{\partial\varphi}\right) \tag{2}$$

   (a) Using the above form for $\mathbf{L}$, show that $\mathbf{L}^2 = \mathbf{L}\cdot\mathbf{L}$ in spherical coordinates becomes

$$\mathbf{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \tag{3}$$

   (b) What is the eigenvalue of the $\mathbf{L}^2$ operator when acting on $Y_\ell^m$?

   (c) What is the eigenvalue of the $L_z$ operator when acting on $Y_\ell^m$?

   (d) Are the eigenvalues what you expected?

   (e) With some potential $V(r)$ specified, the radial equation will give rise to another quantum number $n$ that gives one energy $E_n$ for every $n$, and the allowed values of $\ell$ for a given $n$ is $\ell = 0, 1 \ldots (n-1)$. One can show that all energies $E_n$ for the different allowed values of $\ell$ and $m$ are equal, or *degenerate*, in the case of a Coulomb potential $V(r) \propto 1/r$. What is the degeneracy of the energy level $E_n$ in terms of $\ell$ and $m$?