# Tutorial Class 8
# Mathematical Methods in Physics

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## 1 The cosmic microwave background radiation

In our current model of the Universe, it started out in an extremely hot and dense state and subsequently expanded and cooled down. A few hundred thousand years after the Big Bang, the early Universe was a hot and dense plasma consisting of photons, nuclei, electrons and dark matter, with atoms completely ionised due to the large thermal energy. When the Universe expanded and cooled down, eventually the thermal energy was not enough to ionise atoms and neutral atoms were formed when electrons combined with nuclei. This is called the *time of recombination* and happened at about 380 000 years after the Big Bang. At this time, since photons were no longer constantly interacting with the nuclei and the electrons in the plasma, they could propagate freely. The corresponding radiation that was emitted then is called the *cosmic microwave background* (CMB) and represents the very first light that we can see in the Universe. The CMB photons were initially at the temperature of the Universe at the time of recombination but have since cooled down to a temperature of 2.72 K.

The temperature of the CMB is *extremely* uniform, the corresponding blackbody spectrum has the most well-defined temperature we have ever observed. However, there are very small fluctuations from the mean value of the temperature on the order of 1 in 100 000. These small fluctuations in fact tell us a lot about the physics in the early Universe when the CMB was emitted.

In the early Universe plasma, regions with higher density acted as gravitational potential wells and tended to attract matter through gravity. However, when matter was pushed together the photons started pushing back, resulting in a radiation pressure out of the potential wells, affecting normal matter (nuclei and electrons). Dark matter on the other hand, just kept contracting since it does not interact with photons, something which tended to make the gravitational potential wells deeper.

In this way, the non-dark matter density oscillated back and forth, resulting in pressure waves, i.e. sound waves. At the time of recombination, the thermal energy of the photons was no longer enough to keep atoms ionised, so neutral atoms formed. These do not interacting to the same extent with the photons since they are electrically neutral, therefore the radiation pressure was no longer effective and the CMB photons could now just travel freely, keeping with them a snapshot of the sound waves at the time of recombination. An observation of the fluctuations in temperature of the CMB photons over the sky therefore maps directly the original overdensities in the early Universe.

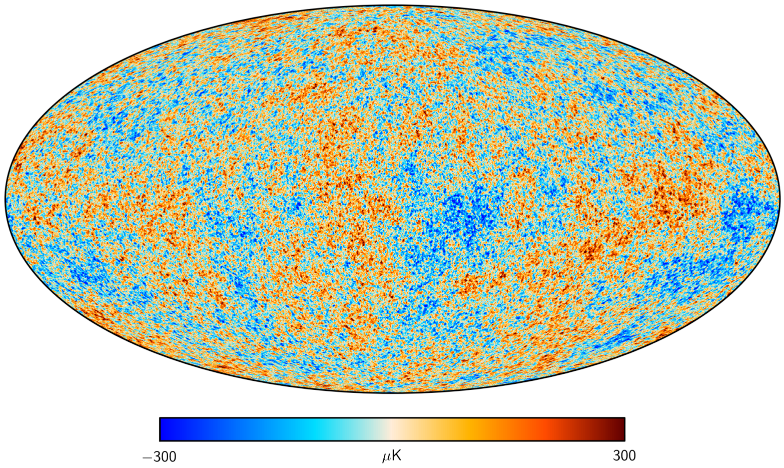


**Figure 1** – The fluctuations in the cosmic microwave background as seen by the Planck telescope. Note that the fluctuations are relatively speaking very small, hundreds of $\mu$K compared to a temperature of 2.78 K, meaning relative fluctuations on the order of .

## 2 The CMB fluctuations and spherical harmonics

In the following, define the *relative fluctuations* from the mean value of the CMB temperature as

$$\delta T(\theta, \varphi) = T(\theta, \varphi) - \langle T \rangle. \tag{1}$$

1. How are the spherical harmonics $Y_\ell^m(\theta, \varphi)$ defined? In what mathematical context do they appear?

2. Write $\delta T(\theta, \varphi)$ as a Laplace series with coefficients $a_{\ell m}$. That is, expand it in spherical harmonics. What are the allowed ranges for $\ell$ and $m$? How many $m$ values are there for a given $\ell$?

3. Show that the $\ell = 0$ term gives the average of $\delta T$ over the whole sky, i.e. show that $a_{00} \propto \langle \delta T(\theta, \varphi) \rangle_{\text{all sky}}$. What should $\langle \delta T(\theta, \varphi) \rangle$ be given how we defined it?

4. Consider $m = 0$. Then there is no $\varphi$-dependence and the $Y_\ell^0(\theta)$ are oscillating functions in $\theta$. How many wavelengths fit between $\theta = 0$ and $\theta = 2\pi$ for a given $\ell$ (i.e. one full revolution)? What is the corresponding wavelength $\lambda_\ell$ and the typical angular size of a fluctuation (half the wavelength)?

5. Write down the orthogonality criterion for the spherical harmonics. Use this to project out the $a_{\ell m}$ in the expansion of $\delta T(\theta, \varphi)$. What is the physical meaning of the $a_{\ell m}$ coefficients?

6. Typically in the analysis of experimental observations, one uses not the $a_{\ell m}$ coefficients[1] but instead the angular power $C_\ell$, which is the average of $|a_{\ell m}|^2$ over $m$. What is $C_\ell$? (It is fine to stop before expanding out the expression for $|a_{\ell m}|^2$.)

[1]The $a_{\ell m}$ themselves are a bad physical observable since they average to zero in an ensemble of universes (the fluctuations are random to begin with so when we average over many different starting conditions/universes the fluctuations will average to zero for every $\ell, m$).

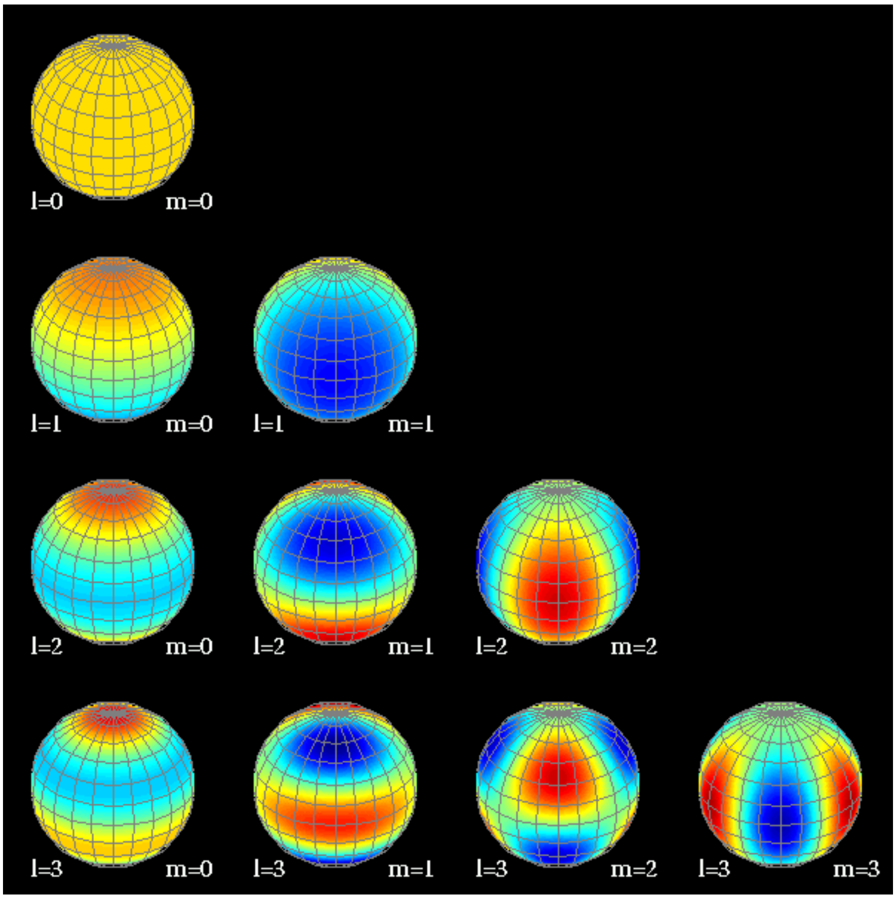


**Figure 2** – The first spherical harmonics projected on a sphere. The color range is $\pm 0.9$.

7. The $\ell = 0$ and $\ell = 1$ terms are usually called the *monopole* and *dipole* terms respectively. The monopole term represents the mean CMB temperature and the dipole term is dominated by the movement of the solar system relative to the CMB rest frame. They are typically not included when analysing the temperature fluctuations, why do you think that is?

8. One can show that patches larger than about $1°$ were not in causal contact at the time of recombination, when the CMB was emitted. This means that most parts of the sky were not in causal contact with most other part parts of the sky at that time and should not have interacted at all. Still, the temperature of the CMB is extremely uniform. Can you see why this is a problem? The idea of inflation is the currently most accepted attempt at solving this problem. What is the $\ell$ value that corresponds to an angular size of $\sim 1°$?

## 3 The CMB power spectrum

In Fig. 3 the so called *power spectrum* is plotted. This is a plot where

$$D_\ell^{\mathrm{TT}} = \ell(\ell+1)C_\ell/2\pi \tag{2}$$

is plotted as a function of $\ell$. Large $\ell$ correspond to small angular sizes on the sky and vice versa. If $C_\ell$ is large, this particular $\ell$ mode is particularly prominent in the analysis of the CMB temperature fluctuations. It means that sound waves in the early Universe plasma with the corresponding wavelength $2\pi/\ell$ were at the extrema of oscillation at the time of recombination. Therefore, the first peak represents the mode that just hade time to do one single oscillation before the time of recombination, whereas the higher $\ell$ peaks are overtones of this fundamental mode, corresponding to modes that had time to do more oscillations before the time of recombination.

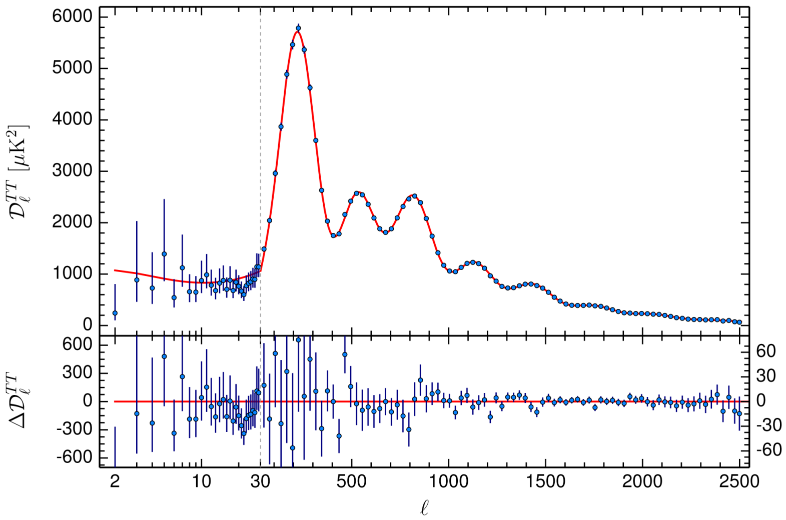


**Figure 3** – The angular power spectrum measured by the Planck telescope. The quantity $D_\ell^{\mathrm{TT}} = \ell(\ell+1)C_\ell/2\pi$ is plotted as a function of $\ell$. Large $\ell$ correspond to smaller angles on the sky (the angular size is $\sim 180°/\ell$). The peaks represents the fundamental mode and overtones of the sound wave oscillations in the early Universe plasma. The red curve is the theoretical prediction. Note that all data points, also at large $\ell$, have error bars.

## 4 Additional reading

Wayne Hu's tutorials at `http://background.uchicago.edu/index.html` (highly recommended). CMB simulators at `https://chrisnorth.github.io/planckapps/Simulator/` or `https://map.gsfc.nasa.gov/resources/camb_tool/index.html`. A tutorial by Clem Pryke at `http://spud.spa.umn.edu/~pryke/logbook/20000922/`.