Green's function for the heat eq^n in $D = 1+n$ dimensions ; $\mathbb{R}^n \times [0, \infty)$ $(t \geq 0)$.

Eq^n: $$\frac{\partial \phi_h(\vec{x},t)}{\partial t} = K \nabla^2 \phi_h(\vec{x},t)$$

(~~As~~ Hom. eq^n, <u>no</u> driving at the moment.

Take first <u>inhom.</u> BC's: $\phi_h(\vec{x}, 0) = f(\vec{x})$, and

$$\lim_{|x| \to \infty} \phi_h(\vec{x}, t) = 0 \quad \forall t$$

Solve the system using a FT w.r.t. the ~~sta~~ spatial variables:

$$\begin{cases} \frac{\partial}{\partial t} \hat{\Phi}_h(\vec{k}, t) = -K |\vec{k}|^2 \hat{\Phi}_h(\vec{k}, t) & (\text{write } |\vec{k}|^2 = k^2) \\ \hat{\Phi}_h(\vec{k}, 0) = \hat{f}(\vec{k}) & (\text{F.T. of } f(\vec{x})) \end{cases}$$

We easily find the solution: $\hat{\phi}_h(\vec{k}, t) = \hat{f}(\vec{k}) e^{-Kk^2 t}$

So, $\hat{\Phi}_h(\vec{x}, t)$ is the inverse transform of a product of two fourier transforms, so $\hat{\Phi}_h(\vec{x}, t)$ is a convolution!

The inverse transform of $\tilde{f}(\vec{k})$ is $f(\vec{x})$, the inverse transform of $\tilde{g}(\vec{k}) = e^{-\kappa k^2 t}$ is given by ~~one~~ a gaussian.

$$\left[\ \tilde{g}(\vec{k}) = e^{-\frac{k^2}{2a}} \rightsquigarrow g(\vec{x}) = a^{n/2} e^{-\frac{ax^2}{2}}\ \right]$$

So, in our case $g(\vec{x},t) = (2\kappa t)^{-n/2} e^{-\frac{x^2}{4\kappa t}}$

~~$\left(a = \frac{1}{2\kappa t}\right)$~~ $\left(a = \frac{1}{2\kappa t}\right)$

Using the convolution thm, we find: $\frac{1}{(2\pi)^{n/2}} \int f(\vec{y})\, g(\vec{x}-\vec{y},t)\, d\vec{y}$

$$\phi_h(\vec{x},t) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(2\kappa t)^{n/2}} \int f(\vec{y})\, e^{-\frac{|\vec{x}-\vec{y}|^2}{4\kappa t}}\, d^n\vec{y}$$

If we take a Gaussian w/ weight Q, i.e.

$$f(\vec{x}) = Q\left(\frac{a}{\pi}\right)^{n/2} e^{-a|\vec{x}|^2}, \quad \int_{-\infty}^{\infty} d^n\vec{x}\ |f(\vec{x})| = Q,$$

results in (little calculation):

$$\phi_h(\vec{x},t) = Q\left(\frac{a/\pi}{1+4a\kappa t}\right)^{n/2} e^{-\left(\frac{a}{1+4\kappa a t}\right)x^2}$$

Limiting case: $f(\vec{x}) = Q\,\delta^{(n)}(\vec{x})$ gives

$\phi_h(\vec{x},t) = \frac{Q}{(2\pi)^{n/2}}\, g(\vec{x},t)$, so δ-pulse at $t=0$ spreads to all x for arbitrary small t. Not relativistic (but causal)

Now: look at driven system, w/ hom. BC's (so opposite from previous case).

$$** \quad \frac{\partial \phi_s(\vec{x},t)}{\partial t} - k\nabla^2 \phi_s(\vec{x},t) = F(\vec{x},t)$$

$$\phi(\vec{x},0) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \phi(\vec{x},t) = 0 \ \forall t$$

So, look at Greens function:

$$g(\vec{x},t;\vec{y},t'), \quad \frac{\partial}{\partial t} g - k\nabla_x^2 g = \delta^n(\vec{x}-\vec{y})\,\delta(t-t'),$$

st $g(\vec{x},0,\vec{y},t') = 0$

Solution to (**) is $\phi_s(\vec{x},t) = \int_0^\infty dt' \int d^n\vec{y}\, g(\vec{x},t,\vec{y},t')\, F(\vec{y},t')$

We take a FT w.r.t. $\vec{x}$;

$$\frac{\partial}{\partial t}\hat{g}(\vec{k},t,\vec{y},t') + k^2 \kappa\, \hat{g}(\vec{k},t,\vec{y},t') = \frac{e^{i\vec{k}\cdot\vec{y}}}{(2\pi)^{n/2}}\,\delta(t-t')$$

Multiply by $e^{\kappa k^2 t}$, which gives

$$\frac{\partial}{\partial t} e^{\kappa k^2 t}\hat{g}(\vec{k},t,\vec{y},t') = \frac{e^{i\vec{k}\cdot\vec{y}+\kappa k^2 t}}{(2\pi)^{n/2}}\,\delta(t-t'),$$

where $\hat{g}(\vec{k},0,\vec{y},t') = 0$, so we can integrate:

$$\hat{g}(\vec{k},t,\vec{y},t') = e^{-\kappa k^2 t}\cdot\frac{e^{i\vec{k}\cdot\vec{y}}}{(2\pi)^{n/2}}\int_0^t e^{\kappa k^2 s}\,\delta(s-t')\,ds$$

$$= \begin{cases} 0 & t<t' \\ \dfrac{e^{i\vec{k}\cdot\vec{y}}}{(2\pi)^{n/2}}\, e^{-\kappa k^2 (t-t')} & t>t' \end{cases}$$

$$= \frac{u(t-t')}{(2\pi)^{n/2}} e^{i\vec{k}\cdot\vec{y} - k k^2 (t-t')}$$

↑ step function

We can now take the inverse transform:

$$g(\vec{x}, t; \vec{y}, t') = \frac{u(t-t')}{(2\pi)^{n}} \int d^n k \, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} e^{-k k^2 (t-t')}$$

This is the inverse transform of the Gaussian we saw in the hom. heat eq^n at shifted variables; namely $g(\vec{x}-\vec{y}, t-t')$, so

$$g(\vec{x}, t; \vec{y}, t') = \frac{u(t-t')}{(2\pi)^{n/2}} \frac{1}{(2k(t-t'))^{n/2}} e^{-\frac{|\vec{x}-\vec{y}|^2}{4k(t-t')}}$$

$$= \frac{u(t-t')}{(2\pi)^{n/2}} g(\vec{x}-\vec{y}, t-t')$$

Compare: $\phi_h(\vec{x}, t) = \frac{g(\vec{x}, t)}{(2\pi)^{n/2}}$; same form!

(w/ $\phi_h(\vec{x}, 0) = \delta^n(\vec{x})$)

This is not a coincidence!

$$\phi_h(\vec{x}, t) = \int d^n y \, f(\vec{y}) \frac{g(\vec{x}-\vec{y}, t)}{(2\pi)^{n/2}}, \quad \text{s.t. } \phi_h(\vec{x}, 0) = f(\vec{x}).$$

If we impose $\phi_h(\vec{x}, t') = f(\vec{x})$ instead, we get:
(ie at a later time $t' < t$)

$$\phi_h(\vec{x}, t) = \int d^n y \, f(\vec{y}) \frac{g(\vec{x}-\vec{y}, t-t')}{(2\pi)^{n/2}}$$

So, $\frac{g(\vec{x}-\vec{y}, t-t')}{(2\pi)^{n/2}}$ 'propagates' initial solution over a time $t-t'$

If we look at the forced problem, we can write the solution as (using step-function in time):

$$\phi_s(\vec{x},t) = \int_0^t \left[\int F(\vec{y},t') \cancel{g(\vec{x},t,\vec{y},t')} \frac{}{(2\pi)^{n/2}} g(\vec{x}-\vec{y},t-t') d\vec{y}^n \right] dt'$$

So, we can view the driving at $F(\vec{y},t')$ as an 'initial' condition at time $t=t'$

The integral over t' represents the accumulated effect of all 'initial' conditions at all times $t'<t$.

Step function implements causality. Solution at time t is only influenced by the drive at times before t!

Relation between hom. eq^n w/ inhomogeneous BC's and the inhom. eq^n w/ hom. BC's is called Duhamel's principle.

$$\phi_h(\vec{x},t) = \frac{1}{(4\pi k t)^{n/2}} \int f(\vec{y})\, e^{-\frac{|\vec{x}-\vec{y}|^2}{4kt}}\, d^n\vec{y}$$

$$f(\vec{x}) = Q\left(\frac{a}{\pi}\right)^{n/2} e^{-a|\vec{x}|^2} \quad ; \int d^n\vec{x}\, f(\vec{x}) = Q$$

We need to calculate $\int e^{-a|\vec{y}|^2} e^{-b|\vec{x}-\vec{y}|^2} d^n\vec{y}$

where $b = \frac{1}{4kt}$

$\cancel{\int e^{-a|\vec{y}|^2} e^{-b|\vec{x}|^2 + 2}}$

Do 1-d. integral first: $e^{-ay^2 - bx^2 + 2bxy - by^2}$

$$= e^{-y^2(a+b) + 2byx}\, e^{-bx^2}$$

$$= e^{-(a+b)\left(y - \frac{bx}{(a+b)}\right)^2 + \frac{b^2x^2}{(a+b)} - \frac{b(a+b)x^2}{(a+b)}}$$

$$= e^{-(a+b)\left(y-\frac{bx}{(a+b)}\right)^2 - \frac{ab}{(a+b)}x^2}$$

1-d int $\int e^{-(a+b)\left(y-\frac{bx}{a+b}\right)^2} e^{-\frac{ab}{(a+b)}x^2} dy = \left(\frac{\pi}{a+b}\right)^{\frac{1}{2}} e^{-\left(\frac{ab}{a+b}\right)x^2}$

n-dim integ: $\left(\frac{\pi}{a+b}\right)^{n/2} e^{-\left(\frac{a}{1+a/b}\right)|\vec{x}|^2}$

So: $\phi_{h}(\vec{x},t) = \left(\frac{1}{\pi}\right)^{n/2} \cancel{\frac{1}{b}}\, b^{n/2} \left(\frac{\pi}{a+b}\right)^{n/2} Q\left(\frac{a}{\pi}\right)^{n/2} e^{-\left(\frac{a}{1+a/b}\right)|\vec{x}|^2}$

$$= Q\left(\frac{a}{(1+a/b)\pi}\right)^{n/2} e^{-\left(\frac{a}{1+a/b}\right)|\vec{x}|^2}$$

$$= Q\left(\frac{a}{\pi + 4kt\pi a}\right)^{n/2} e^{-\left(\frac{a}{1+4kta}\right)x^2}$$