Method of characteristics

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1 For quasi-linear first order PDEs

1.1 General theory

Let $u(x,y)$ be a function of two independent variables solution of the following :

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u) \tag{1}$$

with the **curve** $\gamma(t)$ in the **space** (x,y,u) as initial condition.

Note : this initial condition is equivalent to specify the function u and its normal derivative $\frac{\partial u}{\partial n}$ along a **curve** Γ in the **plane** (x,y).

The solution of equation (1) is a **surface** $\mathcal{S}$ with cartesian equation $u(x,y) - u = 0$ in the **space** (x,y,u). It is well-known that a surface $S(x,y,u) = 0$ has a normal vector field $\vec{\nabla}\, S$. Hence $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right)^t$ is normal to $\mathcal{S}$ at every point. Furthermore, we notice :

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a(x,y,u) \\ b(x,y,u) \\ c(x,y,u) \end{pmatrix} = 0 \tag{2}$$

We have just made sure that the vector field $\left(a,b,c\right)^t$ is tangent to $\mathcal{S}$ at every point. **Field lines** of this vector field will be called **characteristic curves**. They are determined by :

$$\begin{pmatrix} dx \\ dy \\ du \end{pmatrix} \times \begin{pmatrix} a(x,y,u) \\ b(x,y,u) \\ c(x,y,u) \end{pmatrix} = \vec{0} \quad \Leftrightarrow \quad \frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \tag{3}$$

We introduce a parametrization of characteristic curves so that the system (3) becomes :

$$\begin{cases} \frac{dx}{ds} = a(x,y,u) \\ \frac{dy}{ds} = b(x,y,u) \\ \frac{du}{ds} = c(x,y,u) \end{cases} \tag{4}$$

We still need to distinguish the different characterics within the familly determined by this autonomous closed system of ODEs. To do it, we use the initial condition $\gamma(t)$. Indeed, when solving the system (4) one gets integration constants and we fix them such that every characteristics crosses the curve $\gamma(t)$ at "time" $s = 0$. In the end, we obtain a complete parametrization of the family of characterics by parameters s and t :

$$\begin{cases} x(s,t) \\ y(s,t) \\ u(s,t) \end{cases} \tag{5}$$

Then, provided that it is possible[1] to express s and t as functions of x and y, we finally get $u(x,y)$.

[1]The jacobian $\frac{\partial(x,y)}{\partial(s,t)}$ does not vanish.

Characteristics are completely determined by the vector field $\big(a, b, c\big)^t$. The existence of solution $u(x, y)$ depends on the initial condition $\gamma(t)$. It is straightforward to see that there is no solution whenever $\gamma(t)$ is along a characteristics or crosses a characteristics more than one time.

After the initial condition $u\big(x(t), y(t)\big) \equiv \gamma(t)$,

$$\frac{du}{dt} = \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{dy}{dt}\frac{\partial u}{\partial y} \tag{6}$$

Note that $\big(x(t), y(t)\big)$ is a parametrization of the curve Γ in the plane (x, y).
Moreover, equation (1) holds along this curve by hypothesis. Then partial derivatives of u satisfies the following linear system :

$$\begin{cases} a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u) \\ \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{dy}{dt}\frac{\partial u}{\partial y} = \frac{du}{dt} \end{cases} \tag{7}$$

It is solvable if, and only if, the determinant

$$\begin{vmatrix} a(x, y, u) & b(x, y, u) \\ \frac{dx}{dt} & \frac{dy}{dt} \end{vmatrix} = a(x, y, u)\frac{dy}{dt} - b(x, y, u)\frac{dx}{dt} \tag{8}$$

is non-zero. There are some curves in the plane (x, y) such that this determinant vanishes along them :

$$a(x, y, u)dy - b(x, y, u)dx = 0 \quad \Leftrightarrow \quad \frac{dx}{a} = \frac{dy}{b} \tag{9}$$

We find back a part of the system (3). In Sommerfeld's book, those curves in the plane (x, y) are the characteristics. They don't lie in the same space than the one defined above and parametrized by (5). If the initial condition Γ is one of these curves, then the system (7) is not solvable and there is no solution to the equation (1) as we have ever predicted. We can actually relate this issue to the question of invertibility of $x(s, t)$ and $y(s, t)$. Indeed, plugging the two first equations of (4) in (8) yields :

$$\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} \\ \frac{dx}{dt} & \frac{dy}{dt} \end{vmatrix} \equiv \frac{\partial(x, y)}{\partial(s, t)} \tag{10}$$

which is exactly the jacobian of the transformation.
Very often, equation (9) can be expressed as the differential of a function ψ. Then, the cartesian equation of charactetistics curves in Sommerfeld's sense is $\psi(x, y) = cst$.

Roughly speaking when everything goes well, every "initial point" of the curve $\gamma(t)$ generates a characteristics and the collection of these curves gives the surface $\mathcal{S}$ that we are looking for.

1.2 Particular cases

Let us consider the case when $a(x, y, u)$ and $b(x, y, u)$ only depend on u and $c(x, y, u) = 0$, *i.e.*

$$a(u)\frac{\partial u}{\partial x} + b(u)\frac{\partial u}{\partial y} = 0 \tag{11}$$

Inspired by Arfken (for whom characteristics have the same sense as Sommerfeld), we perform the change of variables :

$$\begin{cases} s = a(u)\ x + b(u)\ y \\ t = b(u)\ x - a(u)\ y \end{cases} \tag{12}$$

Keeping in mind that a and b do not depend on x and y, equation (11) becomes :

$$a(u)\left(a(u)\frac{\partial u}{\partial s} + b(u)\frac{\partial u}{\partial t}\right) + b(u)\left(b(u)\frac{\partial u}{\partial s} - a(u)\frac{\partial u}{\partial t}\right) = \Big(a^2(u) + b^2(u)\Big)\ \frac{\partial u}{\partial s} = 0 \tag{13}$$

whose general solution is obviously $u(x, y) = f(t) = f\big(b(u)\ x - a(u)\ y\big)$, which is an implicit equation.
In this particular case, equation (9) is simple so that we readily find the above-mentionned function ψ :

$$b(u)\ dx - a(u)\ dy = d\big(b(u)\ x - a(u)\ y\big) = 0 \quad \Leftrightarrow \quad b(u)\ x - a(u)\ y = t = cst \tag{14}$$

Thus, the characteristics in Sommerfeld's sense are the straight lines $t = cst$ *viz.* the solution $u(x, y)$ is constant along its characteristics.

There is actually a more general case in which the solution $u(x, y)$ is constant along its characteristics : it is sufficient to require $c(x, y, u) = 0$ in equation (1). Indeed, equation of characteristics in the plane (x, y) is :

$$\frac{dy}{dx} = \frac{b(x, y, u)}{a(x, y, u)} \tag{15}$$

Then, divide equation (1) by $a(x, y, u)$ to identify this term and eventually obtain :

$$\frac{\partial u}{\partial x} + \frac{dy}{dx}\,\frac{\partial u}{\partial y} \equiv \frac{du\big(x, y(x)\big)}{dx} = 0 \tag{16}$$

We end this section with two classical examples from physics. First one is the simplest linear wave equation :

$$c\,\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \tag{17}$$

From our study, we immediately infer the general solution : $u(x, t) = f(x - ct)$ where the function f is given by initial condition $u(x, 0) = f(x)$. Moreover, the perturbation u propagates with velocity c and without deformation along parallel straight lines $x - ct = cst$ as any wave should do.

Second example is a non-accelerating fluid. The velocity field $u\big(x(t), t\big)$ obeys the following equation :

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = 0 \tag{18}$$

However, due to the link between Eulerian and Lagrangian points of vue we identifie $\frac{dx}{dt} \equiv u\big(x(t), t\big)$. In the end, we get a quasi-linear equation :

$$u\,\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \tag{19}$$

The general solution is clearly given by the implicit equation $u(x, t) = f(x - ut)$ where the function f is given by initial condition $u(x, 0) = f(x)$. Characteristics are straight lines but their slope now depends on u so that they are no more parallel : $x - ut = cst$. It means that a fluid particle with velocity u moves all along a given characteristic at this velocity u. Subsequently, after a while the fast-moving fluid particles catch the slow-moving ; that occurs at the crossing point of their characteristics. It entails a serious issue of interpretation since the velocity field gets multivalued. In the real world, it leads to the wave breaking. This issue is mathematically solved by introducing dispersive (Korteveg-De Vries equation) or dissipative (Burgers equation) terms.

2 For hyperbolic PDEs

In this section, we generalize the method of characteristics to a special class of second order PDEs : hyperbolic equations. Their general form is :

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + 2B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} = \phi\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, x, y\right) \quad \text{with} \quad B^2 - AC > 0 \tag{20}$$

We follow Sommerfeld's approach and denote $p \triangleq \frac{\partial u}{\partial x}$ and $q \triangleq \frac{\partial u}{\partial y}$. Then,

$$\begin{cases} dp = \frac{\partial^2 u}{\partial x^2}\,dx + \frac{\partial^2 u}{\partial x \partial y}\,dy \\ dq = \frac{\partial^2 u}{\partial x \partial y}\,dx + \frac{\partial^2 u}{\partial y^2}\,dy \end{cases} \tag{21}$$

Equations (20) and (21) provide a linear system for second order derivatives of u. Its determinant is :

$$\Delta \triangleq \begin{vmatrix} A(x, y) & 2B(x, y) & C(x, y) \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = A(x, y)\,dy^2 - 2B(x, y)\,dxdy + C(x, y)\,dx^2 \tag{22}$$

We remind that the initial condition can be seen as a curve Γ in the plane (x, y) along which the function u and its normal derivative $\frac{\partial u}{\partial n}$ are specified. Hence, the first order partial derivatives p and q are known along Γ. Beside, we can infer second order derivatives along Γ by solving the above-mentionned linear system and that is possible provided that $\Delta \neq 0$.
It is interesting to notice that we can pursue the procedure and establish another linear system for the third order derivatives of u and so on. Amazingly, all these linear systems have the same determinant Δ. And if it does not vanish, one can compute all partial derivatives of u at any order along Γ. Then Sommerfeld proposed an unusual approach of solving a PDE. The solution of equation (20) is obtained in the neighbourhood of Γ by a Taylor expansion of u using partial derivatives that we have just got.

At this stage, Sommerfeld defines characteristics as the curves of the plane (x, y) along which $\Delta = 0$. It yields the equation of special conic :

$$A(x,y)\, dy^2 - 2B(x,y)\, dxdy + C(x,y)\, dx^2 = 0 \quad \text{with} \quad B^2 - AC > 0 \tag{23}$$

In analytic geometry, one can show that there is a frame where equation (23) has the following form :

$$\left(\frac{dX}{\alpha(X,Y)}\right)^2 - \left(\frac{dY}{\beta(X,Y)}\right)^2 = 0 \quad \Leftarrow \quad \begin{cases} \frac{dX}{\alpha(X,Y)} - \frac{dY}{\beta(X,Y)} = 0 \\ \frac{dX}{\alpha(X,Y)} + \frac{dY}{\beta(X,Y)} = 0 \end{cases} \tag{24}$$

In the end, we have not one but two families of characteristic curves. As previously, very often their cartesian equations can be put on the form $\varphi(x, y) = cst$ and $\psi(x, y) = cst$.

We generalize the geometric condition of solvability of the first-order case. Equation (20) has a solution if we are able to calculate all partial derivatives of u at any order along Γ so as to perform the Taylor expansion, that is if $\Delta \neq 0$, that is if the initial condition Γ is not along a characteristics.

In Sommerfeld's book, it is shown that thanks to the smart change of variables

$$\begin{cases} \xi = \varphi(x,y) + \psi(x,y) \\ \eta = \varphi(x,y) - \psi(x,y) \end{cases} \tag{25}$$

equation (20) has the normal form :

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} = \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) u(x,y) = \tilde{\phi}\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right) \tag{26}$$

This form justifies why equation (20) is called hyperbolic. We notice that it is equivalent to the following system of first order quasi-linear PDEs :

$$\begin{cases} \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) u(x,y) = v(x,y) \\ \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) v(x,y) = \tilde{\phi}\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right) \end{cases} \tag{27}$$

It explains why there are two families of characteristics.

We finally apply this method to D'Alembert wave equation :

$$c^2\, \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad \Leftrightarrow \quad \begin{cases} \left(c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u(x,t) = v(x,t) \\ \left(c\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) v(x,t) = 0 \end{cases} \tag{28}$$

First, it is clear that $v(x, t) = h(x + ct)$ so that we have to solve

$$c\,\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = h(x + ct) \tag{29}$$

When this equation is homogeneous, we have already seen that the solution is of the form $f(x - ct)$. But the inhomogenous part is a function with argument $x + ct$, so we have to keep the homogenous solution and add another function with argument $x + ct$. It is D'Alembert solution :

$$u(x,t) = f(x - ct) + g(x + ct) \tag{30}$$

The function g is the particular solution of equation (**??**) which inhomogenous and subsequently it is complelety determined by h. However, since h is an arbitrary function g is in the end also an arbitrary function.