Legendre polynomials

We obtain the associated Legendre equation when separating equations of the form $-\nabla^2\psi + V(r)\psi = \lambda\psi$ in spherical ~~polar~~ coordinates through $\psi(r,\vartheta,\varphi) = R(r)\Theta(\vartheta)\Phi(\varphi)$

The equation for $\Theta(\vartheta)$: $\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{d\Theta}{d\vartheta}\right) - \frac{m^2}{\sin^2\vartheta}\Theta + l(l+1)\Theta = 0$

Set $x = \cos\vartheta$, $dx = -\sin\vartheta\, d\vartheta$ and $\frac{d}{d\vartheta} = -\sin\vartheta\frac{d}{d(\cos\vartheta)}$; $\frac{1}{\sin\vartheta}\frac{d}{d\vartheta} = -\frac{d}{dx}$

$$\Rightarrow (1-x^2)P'' - 2xP' - \frac{m^2}{1-x^2}P + l(l+1)P = 0$$

The Legendre equation corresponds to $m = 0$.

$$(1-x^2)P'' - 2xP' + \lambda P = 0$$

regular singular points at $x = \pm 1$ and $x = \infty$

A series expansion around $x = 0$ converges for $|x| < 1$ independent of λ. At $|x| = 1$ the expansion diverges unless $\lambda = l(l+1)$ with l an integer. In that case the series terminates and gives polynomials of finite order, $P_l(x)$, which we will focus on.

From chapter 12 we have the generating function

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

We'll use $g(x,t)$ to derive recurrence relations and the scale. To find the scale set $x = 1$:

$$g(1,t) = \frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{\sqrt{(1-t)^2}} = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \Rightarrow \underline{P_n(1) = 1}$$

set $x = -x$, $t = -t$ to find the parity

$$\sum_{n=0}^{\infty} P_n(-x)(-t)^n = \frac{1}{\sqrt{1-2(-x)(-t)+(-t)^2}} = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$$\therefore P_n(-x) = (-1)^n P_n(x) \quad \text{and} \quad P_n(1) = 1 \Rightarrow P_n(-1) = (-1)^n$$

The value at $x=0$. Since $P_{2n+1}(-x) = -P_{2n+1}(x)$ $P_{2n+1}(0)=0$ for odd-index P_n. So $g(0,t) = (1+t^2)^{-1/2} =$

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} t^{2n} = \sum_{n=0}^{\infty} P_{2n}(0)\, t^{2n}$$

The binomial coefficient $\binom{-1/2}{n} = \frac{1}{n!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right) =$

$$= (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!} = (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

so that $P_{2n}(0) = (-1)^n \dfrac{(2n-1)!!}{(2n)!!}$

Do a binomial expansion of the generating function:

$\dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-2xt+t^2)^n$ shows that the leading power of x to multiply t^n is x^n with coefficient $(-2)^n$ so that the coefficient of x^n in $P_n(x)$ is then $\binom{-1/2}{n}(-2)^n = \dfrac{(2n-1)!!}{n!}$

∴ $P_n(x)$ is a polynomial of degree n in x

Problem 15.1.2 A closed expression for the Legendre polynomials

We have $\dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-2xt+t^2)^n = \sum_{n=0}^{\infty} (-1)^n \dfrac{(2n-1)!!}{(2n)!!} (-2xt+t^2)^n$

write $\dfrac{(2n-1)!!}{(2n)!!} = \dfrac{2^n \cdot n!\,(2n-1)!!}{2^n\cdot n!\; 2^n\cdot n!} = \dfrac{(2n)!}{2^{2n}(n!)^2}$ and do a binomial expansion of the parenthesis $(-2xt+t^2)^n = t^n(-2x+t)^n$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}\, t^n \sum_{k=0}^{n} \frac{(-1)^{n-k}\, n!}{k!\,(n-k)!} (2x)^{n-k}\cdot t^k =$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k (2n)!}{2^{2n}\, n!\, k!\,(n-k)!} (2x)^{n-k}\, t^{n+k}$$

Rearrange the summation according to $m = n+k$ $0 \le m \le \infty$ and $n = m-k$ $\rightarrow (2x)^{m-2k}$ where $0 \le k \le [m/2]$ with $[m/2]$ the integer part i.e. $[m/2] = m/2$ for m even and $(m-1)/2$ for m odd (see (1.31) in A,W,H and (5.64) in A&W 6th)

So $$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{m=0}^{\infty} \sum_{k=0}^{[m/2]} (-1)^k \frac{(2m-2k)!}{2^{2m-2k}\, k!\,(m-k)!\,(m-2k)!}\, 2^{m-2k} x^{m-2k} t^m \quad 3$$

$$= \sum_{m=0}^{\infty} t^m \sum_{k=0}^{[m/2]} (-1)^k \frac{(2m-2k)!}{2^m\, k!\,(m-k)!\,(m-2k)!}\, x^{m-2k} \equiv \sum_{m=0}^{\infty} t^m P_m(x)$$

$\therefore$ $P_n(x)$ contains alternate powers of x: $x^n, x^{n-2}, x^{n-4}, \dots (x^0 \text{ or } x)$

$\Rightarrow$ $P_0(x) = 1$ $\quad (P_n(1) = 1)$ and $P_1(x) = x$

Recurrence formulas

Differentiate $g(x,t)$ with respect to t:

$$\frac{\partial g(x,t)}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1}$$

Rearrange to $\quad (\cancel{t}-\cancel{x}) \sum_{n=0}^{\infty} P_n(x) t^n + (1-2xt+t^2) \sum_{n=0}^{\infty} n P_n(x) t^{n-1} = 0$

Expand $$\sum_{n=0}^{\infty} P_n(x) t^{n+1} - \sum_{n=0}^{\infty} x P_n(x) t^n + \sum_{n=0}^{\infty} n P_n(x) t^{n-1} - 2\sum_{n=0}^{\infty} n x P_n(x) t^n +$$
$$+ \sum_{n=0}^{\infty} n P_n(x) t^{n+1} = 0$$

Collect terms t^n: $$\sum_{n=0}^{\infty} \left\{ P_{n-1}(x) - x P_n(x) + (n+1) P_{n+1}(x) - 2n x P_n(x) + (n-1) P_{n-1}(x) \right\} t^n = 0$$

$$\Rightarrow \boxed{(2n+1) x P_n(x) = n P_{n-1}(x) + (n+1) P_{n+1}(x)}$$

With $P_0(x) = 1$ and $P_1(x) = x$ we obtain

$$2 P_2(x) = 3x^2 - 1 \quad \rightarrow \quad P_2(x) = \frac{1}{2}(3x^2 - 1) \quad \text{etc}$$

Differentiate with respect to x:

$$\frac{\partial}{\partial x} g(x,t) = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x) t^n$$

$$\Rightarrow (1-2xt+t^2) \sum_{n=0}^{\infty} P_n'(x) t^n - t \sum_{n=0}^{\infty} P_n(x) t^n = 0$$

$$\sum_{n=0}^{\infty} P_n'(x) t^n - 2\sum_{n=0}^{\infty} x P_n'(x) t^{n+1} + \sum_{n=0}^{\infty} P_n'(x) t^{n+2} - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = 0$$

t^{n+1}: $$\boxed{P_{n+1}'(x) + P_{n-1}'(x) = 2x P_n'(x) + P_n(x)}$$

Exercise 15.1.1. Derive the equation from the recurrence relations

Differentiate $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$

$$\Rightarrow (2n+1)P_n + (2n+1)xP_n' = (n+1)P_{n+1}' + nP_{n-1}'$$

Subtract

$$P_{n+1}' + P_{n-1}' = 2xP_n' + P_n$$

$$-\quad (2n+1)P_n = -P_{n+1}' + P_{n-1}'$$

$$P_{n+1}' + P_{n-1}' - (2n+1)P_n = 2xP_n' + P_n - P_{n+1}' + P_{n-1}'$$

$$\Rightarrow 2P_{n+1}' = 2(n+1)P_n + 2xP_n'$$

$$\Rightarrow \boxed{P_{n+1}' = (n+1)P_n + xP_n'}$$

Add

$$P_{n+1}' + P_{n-1}' = 2xP_n' + P_n$$

$$(2n+1)P_n = P_{n+1}' - P_{n-1}'$$

$$2P_{n-1}' = -2nP_n + 2xP_n' \Rightarrow \boxed{P_{n-1}' = -nP_n + xP_n'}$$

In $P_{n+1}' = (n+1)P_n + xP_n'$ shift n to $n-1$

$$\boxed{P_n' = nP_{n-1} + xP_{n-1}'} \quad (*)$$

scale $P_{n-1}' = -nP_n + xP_n'$ by x to eliminate P_{n-1}' in $(*)$

$$P_n' = nP_{n-1} - nxP_n + x^2P_n' \Rightarrow \boxed{(1-x^2)P_n' = nP_{n-1} - nxP_n}$$

Differentiate: $\frac{d}{dx}\left[(1-x^2)P_n'\right] = \frac{d}{dx}\left[nP_{n-1} - nxP_n\right]$

$$\Rightarrow (1-x^2)P_n'' - 2xP_n' = nP_{n-1}' - nP_n - nxP_n'$$

Eliminate P_{n-1}' using $P_{n-1}' = -nP_n + xP_n'$

$$\Rightarrow (1-x^2)P_n'' - 2xP_n' = -n^2P_n + nxP_n' - nP_n - nxP_n' = -n(n+1)P_n$$

$$\therefore \boxed{(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0}$$

Rodrigues formula for the Legendre polynomials

For a self-adjoint ODE we can write the solutions as $y_n(x) = \frac{1}{w(x)}\left(\frac{d}{dx}\right)^n \left[w\, p(x)^n\right]$

The Legendre equation $(1-x^2)P_n'' - 2x P_n' + n(n+1) P_n = 0$ is already self-adjoint so $w(x) = 1$ and $p(x) = 1-x^2$

So $P_n(x) \sim \left(\frac{d}{dx}\right)^n (1-x^2)^n$. We need to establish the scale. The leading term (highest power) will be x^n.

Consider the x^n term in the series expansion

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!} x^{n-2k}$$ which is found for $k=0$ and is $\frac{(2n)!}{2^n\, n!\, n!} x^n$; From the Rodrigues formula we find $P_n(x) = C_n \left(\frac{d}{dx}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^{2n-2k}$

The highest power x^n has the coefficient $(k=0)$

$C_n\, (-1^n) \frac{(2n)!}{n!}$ so that with $C_n = \frac{(-1)^n}{2^n\, n!}$ we have the correct scaling and $P_n(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n (1-x^2)^n =$

$$= \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

Orthogonality and normalization

Since the Legendre ODE is self-adjoint and $(1-x^2)$ as coefficient of $P_n''(x)$ vanishes at $x = \pm 1$ the solutions with different index will automatically be orthogonal with unit weight over $[-1,1]$.

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \,, \quad n \neq m$$

To find the normalization we square the equation for the generating function:

$$\frac{1}{1-2xt+t^2} = \left[\sum_{n=0}^{\infty} P_n(x)\, t^n\right]^2$$

Integrate from $x=-1$ to $x=1$ and use the orthogonality

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\int_{-1}^{1} P_n(x)P_m(x)\,dx \underset{\text{orthogonality}}{=} \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}\left[P_n(x)\right]^2 dx$$

Substitute $y = 1-2xt+t^2$, $dy = -2t\,dx$ in the integral to the left

$x=-1 \rightarrow y=(1+t)^2$

$x=1 \rightarrow y=(1-t)^2$

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = -\frac{1}{2t}\int_{(1+t)^2}^{(1-t)^2}\frac{dy}{y} =$$

$$= \frac{1}{2t}\int_{(1-t)^2}^{(1+t)^2}\frac{dy}{y} = \frac{1}{t}\cdot\frac{1}{2}\left\{2\ln(1+t) - 2\ln(1-t)\right\} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$$

Make a power series expansion (Exercise 1.6.1)

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots$$

$$\ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \ldots$$

so that $\ln\left(\frac{1+t}{1-t}\right) = 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \ldots\right) = 2t\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1}$

and $\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = 2\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1} = \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}\left[P_n(x)\right]^2 dx$

$$\therefore \quad \underline{\underline{\int_{-1}^{1}\left[P_n(x)\right]^2 dx = \frac{2}{2n+1}}}$$

and $$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\,\delta_{nm}$$

Legendre series

With the orthonormality of the Legendre functions we can use them as a basis for expanding other functions defined on the range $[-1,1]$ as $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$

where the coefficients a_n are obtained through projection

$$\sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x) P_m(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{2}{2m+1}\delta_{nm} = a_m \cdot \frac{2}{2m+1} =$$

$$= \int_{-1}^{1} f(x) P_m(x)\,dx \quad \Rightarrow a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\,dx$$

The Laplace equation separated in spherical polar coordinates (spherical symmetry) has the general solution:

$$\psi(r,\vartheta,\varphi) = \sum_{l,m} (A_{lm} r^{l} + B_{lm} r^{-l-1}) P_l^m(\cos\vartheta)(A'_{lm}\sin m\varphi + B'_{lm}\cos m\varphi)$$

where l is an integer to avoid divergence for $\vartheta = 0$ or π.

Specialize to $m=0$ (Legendre functions; for $m \neq 0$ we have the associated Legendre functions):

$$\psi(r,\vartheta) = \sum_{l=0}^{\infty} (a_l r^l + b_l r^{-l-1}) P_l(\cos\vartheta)$$

Consider a spherical boundary and the requirement that $\psi(r,\vartheta)$ remain finite. Then

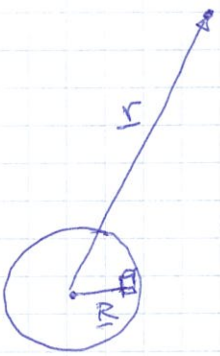
inside the sphere $\psi(r,\vartheta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\vartheta)$

outside the sphere $\psi(r,\vartheta) = \sum_{l=0}^{\infty} a_l r^{-l-1} P_l(\cos\vartheta)$

The Legendre functions describe deviations from spherical symmetry.

Example: Earth's gravitational field

Put the origin at the center of mass of the earth. Let $\underline{R}$ be the position vector of a ~~mass~~ volume element dV in the earth with mass $dm = \rho\, dV$



The contribution to the potential at $\underline{r}$ where $|\underline{r}| \gg |\underline{R}|$ is

$$-\frac{G\,dm}{|\underline{r}-\underline{R}|} = -\frac{G\,dm}{\sqrt{r^2 - 2rR\cos\vartheta + R^2}}$$

$$= -\frac{G\,dm}{r\sqrt{1 - 2\frac{R}{r}\cos\vartheta + \left(\frac{R}{r}\right)^2}} = -\frac{G\,dm}{r}\left\{\sum_{l=0}^{\infty} P_l(\cos\vartheta)\left(\frac{R}{r}\right)^l\right\}$$

The potential at r from the whole mass distribution is

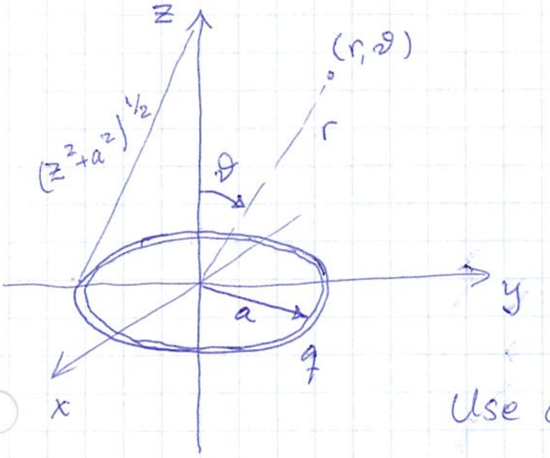
$$U(\underline{r}) = -\frac{G}{r}\sum_{l=0}^{\infty}\int \rho P_l(\cos\vartheta)\left(\frac{R}{r}\right)^l d^3R =$$

$$= -\frac{G}{r}\sum_{l=0}^{\infty}\frac{1}{r^l}\int P_l(\cos\vartheta)\,R^l\,\rho(\underline{R})\,d^3R =$$

$$= -\frac{G}{r}\left\{\underbrace{\int \rho(\underline{R})\,d^3R}_{M} + \frac{1}{r}\underbrace{\int R\cos\vartheta\,\rho(\underline{R})\,d^3R}_{\text{Center of mass}} + \sum_{l=2}^{\infty}\frac{1}{r^l}\int R^l P_l(\cos\vartheta)\,\rho\, dV\right.$$

$$= -\frac{GM}{r} - \sum_{l=2}^{\infty}\frac{G}{r^{l+1}}\underbrace{\int R^l P_l(\cos\vartheta)\,\rho\, dV}_{\text{deviations from spherical symmetry}}$$

Example: Potential from a thin conducting ring of radius a placed symmetrically in the equatorial plane of a spherical polar coordinate system.



The potential Φ satisfies Laplace's equation

Solutions must go to zero as $r \to \infty$ so

same dimension for all terms and c_n

$$\Phi(r,\vartheta) = \sum_{n=0}^{\infty} c_n \frac{a^n}{r^{n+1}} P_n(\cos\vartheta), \quad r > a$$

We need the coefficients c_n

Use Coulomb's law to get the potential at $(z, 0)$

$$\Phi(z,0) = \frac{q}{4\pi\varepsilon_0} \frac{1}{(z^2+a^2)^{1/2}} = \frac{q}{4\pi\varepsilon_0 z} \sum_{s=0}^{\infty} \binom{-1/2}{s} \left(\frac{a^2}{z^2}\right)^s =$$

$$= \frac{q}{4\pi\varepsilon_0 z} \sum_{s=0}^{\infty} (-1)^s \frac{(2s-1)!!}{(2s)!!} \left(\frac{a}{z}\right)^{2s}, \quad z > a$$

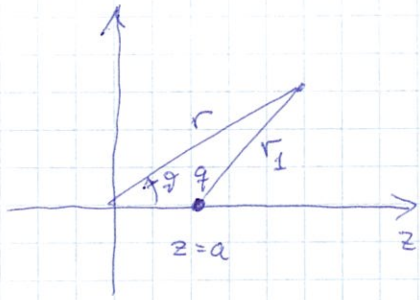
$$\text{But } \Phi(z,0) = \sum_{n=0}^{\infty} c_n \frac{a^n}{z^{n+1}} P_n(1) = \sum_{n=0}^{\infty} \frac{1}{z} \cdot c_n \left(\frac{a}{z}\right)^n$$

so that for odd n, $c_n = c_{2k+1} = 0$

for n even $c_n = c_{2s} = (-1)^s \dfrac{(2s-1)!!}{(2s)!!}$

$$\text{and } \Phi(r,\vartheta) = \frac{q}{4\pi\varepsilon_0 r} \sum_{s=0}^{\infty} (-1)^s \frac{(2s-1)!!}{(2s)!!} \left(\frac{a}{r}\right)^{2s} P_{2s}(\cos\vartheta) \quad r > a$$

Multipole expansion



The potential at (r,ϑ) from a charge q at ~~z~~ $=a$

$$\Psi(r,\vartheta) = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r_1} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{|\underline{r}-\underline{a}|} =$$

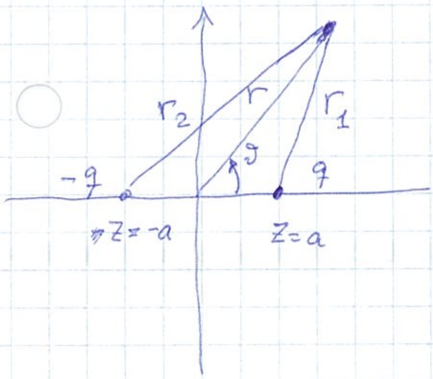
$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{\sqrt{r^2+a^2-2ar\cos\vartheta}} = \frac{q}{4\pi\varepsilon_0 r}\left\{1-2\frac{a}{r}\cos\vartheta+\left(\frac{a}{r}\right)^2\right\}^{-\frac{1}{2}}$$

$r > a$

$$= \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\vartheta)\left(\frac{a}{r}\right)^n =$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\left\{\underline{\frac{1}{r}} + \sum_{n=1}^{\infty} P_n(\cos\vartheta)\left(\frac{a}{r}\right)^n\right\}$$ leading term monopole

Cancel the monopole contribution by adding counter charge:



Now $\Psi(r,\vartheta) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) =$

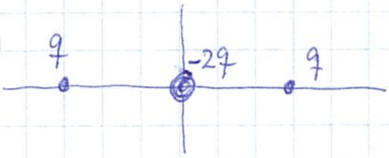
$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{r}-\underline{a}|}-\frac{1}{|\underline{r}+\underline{a}|}\right) = \frac{q}{4\pi\varepsilon_0 r}\left\{\left(1-2\frac{a}{r}\cos\vartheta+\left(\frac{a}{r}\right)^2\right)^{-1/2} - \left(1+2\frac{a}{r}\cos\vartheta+\left(\frac{a}{r}\right)^2\right)^{-1/2}\right\} =$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left\{\sum_{n=0}^{\infty} P_n(\cos\vartheta)\left(\frac{a}{r}\right)^n - \sum_{n=0}^{\infty} P_n(\cos\vartheta)\left(-\frac{a}{r}\right)^n\right\} =$$

$$= \frac{2q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_{2n+1}(\cos\vartheta)\left(\frac{a}{r}\right)^{2n+1}$$ with leading term

the dipole contribution $\frac{2q}{4\pi\varepsilon_0}\,\frac{a P_1(\cos\vartheta)}{r^2} =$

$$= \frac{2a\cdot q}{4\pi\varepsilon_0}\,\frac{P_1(\cos\vartheta)}{r^2}$$ set $2a\cdot q$ (separation × charge) equal to μ the dipole moment



$\sum q_i = 0 \rightarrow$ no monopole

$\sum \mu_i = 0 \rightarrow$ no dipole

Leading term is quadrupole $\frac{\mu_2 P_2(\cos\vartheta)}{r^3}$

Always applicable to inverse-square forces, i.e. where potential depends on $\frac{1}{|\underline{r}_1-\underline{r}_2|}$

Associated Legendre Equation

We determine the eigenvalue λ and form of the solutions to

$$(1-x^2)P''(x) - 2xP'(x) + \left[\lambda - \frac{m^2}{1-x^2}\right]P(x) = 0$$

Substitute $P = (1-x^2)^{m/2}\mathcal{P}$

$$P' = (1-x^2)^{m/2}\mathcal{P}' - mx(1-x^2)^{\frac{m}{2}-1}\mathcal{P}$$

$$P'' = (1-x^2)^{m/2}\mathcal{P}'' - 2mx(1-x^2)^{\frac{m}{2}-1}\mathcal{P}' +$$

$$+ \left\{-m(1-x^2)^{\frac{m}{2}-1} + mx^2(m-2)(1-x^2)^{\frac{m}{2}-2}\right\}\mathcal{P}$$

So that $(1-x^2)P'' - 2xP' + \left[\lambda - \frac{m^2}{1-x^2}\right]P =$

$$= (1-x^2)^{\frac{m}{2}+1}\mathcal{P}'' - 2mx(1-x^2)^{\frac{m}{2}}\mathcal{P}' + \left\{-m(1-x^2)^{\frac{m}{2}} + (m^2-2m)x^2(1-x^2)^{\frac{m}{2}-1}\right\}\mathcal{P}$$

$$- 2x(1-x^2)^{m/2}\mathcal{P}' + 2mx^2(1-x^2)^{\frac{m}{2}-1}\mathcal{P} + \left[\lambda - \frac{m^2}{1-x^2}\right](1-x^2)^{m/2}\mathcal{P} = 0$$

Eliminate $(1-x^2)^{m/2}$

$$(1-x^2)\mathcal{P}'' - \{2mx + 2x\}\mathcal{P}' + \left[-m + \frac{(m^2-2m)x^2}{1-x^2} + \frac{2mx^2}{1-x^2} + \right.$$

$$\left. + \lambda - \frac{m^2}{1-x^2}\right]\mathcal{P} =$$

$$= (1-x^2)\mathcal{P}'' - 2x(m+1)\mathcal{P}' + [\lambda - m(m+1)]\mathcal{P} = 0$$

Make the ansatz $\sum_j a_j x^{k+j} = \mathcal{P}(x)$

$$\mathcal{P}' = \sum_j (k+j)a_j x^{k+j-1} \quad ; \quad \mathcal{P}'' = \sum_j (k+j)(k+j-1)a_j x^{k+j-2}$$

$$(1-x^2)\sum_j (k+j)(k+j-1)a_j x^{k+j-2} - 2x(m+1)\sum_j (k+j)a_j x^{k+j-1} + [\lambda - m(m+1)]\sum_j a_j x^{k+j} =$$

$$= \sum_j (k+j)(k+j-1)a_j x^{k+j-2} + \sum_j a_j x^{k+j}\{-(k+j)(k+j-1) - 2(m+1)(k+j) +$$

$$+ [\lambda - m(m+1)]\} = 0$$

indicial equation for leading term x^{k-2}: $k(k-1) = 0 \Rightarrow k = \begin{cases} 0 \\ 1 \end{cases}$

for $k=0$: $$\sum_j x^j\left\{a_{j+2}(j+2)(j+1) + \{\lambda - m(m+1) \underbrace{- j(j-1) - 2j(m+1)}_{-j^2-(2m+1)j}\}a_j\right\}$$

$$\therefore \quad a_{j+2} = \frac{j^2 + (2m+1)j - \lambda + m(m+1)}{(j+1)(j+2)}a_j$$

But $\left|\frac{a_{j+2}}{a_j}\right| \to 1$ for $j \to \infty$ so the series will in general diverge at $x = \pm 1$ unless it terminates. For $\lambda = \ell(\ell+1)$ we have

$$j^2 + (2m+1)j - \ell(\ell+1) + m(m+1) = 0$$ which is satisfied for

$\ell \geq m$ $\quad j = \ell - m \quad$ (Check: $(\ell-m)^2 + (2m+1)(\ell-m) = \ell^2 - \cancel{2m\ell} + m^2 + \cancel{2m\ell} - 2m^2 + \ell - m = \ell(\ell+1) - m(m+1)$ ✓)

$\ell, m \geq 0$

We find that the regular solutions to the associated Legendre equation depend on two indices, ℓ and m. The solutions are denoted $P_\ell^m(x)$ and we have

$$P_\ell^m(x) = (1-x^2)^{m/2} \mathcal{P}_\ell^m(x)$$ with $\mathcal{P}_\ell^m(x)$ a polynomial of degree $\ell - m$

Generate the equation from Legendre ~~polynomials~~ equation

Apply Leibniz's formula for differentiating a product m times to the Legendre equation

$$\frac{d^m}{dx^m}\left[A(x)B(x)\right] = \sum_{s=0}^{m} \binom{m}{s} \frac{d^{m-s}A(x)}{dx^{m-s}} \frac{d^s B(x)}{dx^s}$$

$$\frac{d^m}{dx^m}\left\{(1-x^2)P_\ell'' - 2xP_\ell' + \ell(\ell+1)P_\ell\right\} = 0$$

set $u \equiv \frac{d^m}{dx^m} P_\ell(x)$

$$\frac{d^m}{dx^m}(1-x^2)P_\ell'' = \binom{m}{0}(1-x^2)\frac{d^m}{dx^m}P_\ell'' + \binom{m}{1}\frac{d}{dx}(1-x^2)\frac{d^{m-1}}{dx^{m-1}}P_\ell'' +$$

$$+ \binom{m}{2}\frac{d^2}{dx^2}(1-x^2)\frac{d^{m-2}}{dx^{m-2}}P_\ell'' = (1-x^2)u'' - 2mx\,u' -$$

$$- \cancel{2} \cdot \frac{m(m-1)}{\cancel{2}} u$$

$$\frac{d^m}{dx^m}(-2x)P_\ell' = \binom{m}{0}(-2x)\frac{d^m}{dx^m}P_\ell' + \binom{m}{1}\left(\frac{d}{dx}(-2x)\right)\frac{d^{m-1}}{dx^{m-1}}P_\ell' =$$

$$= -2x\,u' - 2m\,u$$

$\therefore$

$$(1-x^2)u'' - 2mxu' - m(m-1)u - 2xu' - 2mu + \ell(\ell+1)u =$$

$$= (1-x^2)u'' - 2(m+1)xu' + [\ell(\ell+1) - m(m+1)]u = 0$$

$\therefore$ The same equation as above if $\lambda = \ell(\ell+1)$

Thus we have $\mathcal{P}_\ell^m(x) = (-1)^m \dfrac{d^m}{dx^m} P_\ell(x)$

convention

$\mathcal{P}_\ell(x)$ polynomials of degree ℓ so $m \leq \ell$

and $P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_\ell(x)$

We have $P_\ell(x) = \dfrac{1}{2^\ell \ell!} \dfrac{d^\ell}{dx^\ell}(x^2-1)^\ell$ (Rodrigues' formula)

so $P_\ell^m(x) = \dfrac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2} \dfrac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$

Since m enters as m^2 in the associated Legendre eq. seek the relation between $P_\ell^{-m}(x)$ and $P_\ell^m(x)$

Problem 15.4.3

Write $(x^2-1)^\ell = (x-1)^\ell (x+1)^\ell$ and apply Leibniz's formula

$$P_\ell^{-m}(x) = \frac{(-1)^{-m}}{2^\ell \ell!}(x-1)^{-m/2}(-1)^{-m/2}(x+1)^{-m/2} \cdot \sum_{j=0}^{\ell-m}\binom{\ell-m}{j}\frac{d^j}{d^j}(x-1)^\ell \frac{d^{\ell-m-j}}{dx^{\ell-m-j}}(x+1)$$

$$= \frac{(-1)^{-m}}{2^\ell \ell!}(x-1)^{-m/2}(-1)^{-m/2}(x+1)^{-m/2} \cdot \sum_{j=0}^{\ell-m}\binom{\ell-m}{j}\frac{\ell!}{(\ell-j)!}(x-1)^{\ell-j}\frac{\ell!}{(m+j)!}(x+1)^{m+j}$$

$$= \frac{(-1)^m \cdot (-1)^{m/2}}{2^\ell \ell!}\sum_{j=0}^{\ell-m}\frac{(\ell-m)!\,\ell!\,\ell!\,(x-1)^{\ell-j-m/2}(x+1)^{j+m/2}}{j!\,(\ell-m-j)!\,(\ell-j)!\,(m+j)!}$$

Apply to P_ℓ^m but $(x\pm1)^\ell$ cannot be differentiated more than ℓ times so this must be reflected in the summation

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(x-1)^{m/2}(-1)^{m/2}(x+1)^{m/2}\sum_{j=m}^{\ell}\binom{\ell+m}{j}\frac{\ell!}{(\ell-j)!}(x-1)^{\ell-j}\frac{\ell!}{(j-m)!}(x+1)^{j-m}$$

$$= \frac{(-1)^m(-1)^{m/2}}{2^\ell \ell!}\sum_{j=m}^{\ell}\frac{(\ell+m)!\,\ell!\,\ell!\,(x-1)^{\ell-j+m/2}(x+1)^{j-m/2}}{j!\,(\ell+m-j)!\,(\ell-j)!\,(j-m)!}$$

Change summation $j \to k+m$

$$P_\ell^m(x) = \frac{(-1)^m(-1)^{m/2}}{2^\ell \ell!}\sum_{k=0}^{\ell-m}\frac{(\ell+m)!\,\ell!\,\ell!\,(x-1)^{\ell-k-m/2}(x+1)^{k+m/2}}{(k+m)!\,(\ell-k)!\,(\ell-k-m)!\,(k)!}$$

Compare the expressions for P_ℓ^{-m} and P_ℓ^m

$$\boxed{P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)}$$ i.e. proportional and $-\ell \le m \le \ell$

<u>Generating function for the Legendre polynomials $\mathcal{P}_\ell^m(x)$</u>

Note that $P_\ell^m(x) = (1-x^2)^{m/2}\mathcal{P}_\ell^m(x)$

For the Legendre polynomials we have that

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

The associated Legendre polynomials are given by

$\mathcal{P}_n^m(x) = (-1)^m \frac{d^m}{dx^m} P_n(x)$ so that

$$(-1)^m \frac{d^m}{dx^m}\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty}\mathcal{P}_n^m(x)t^n = (-1)^m\frac{d^m}{dx^m}\frac{1}{(1-2xt+t^2)^{1/2}} =$$

$$= \frac{(-1)^m t^m (2m-1)!!}{(1-2xt+t^2)^{m+1/2}}$$

$$\therefore \quad \frac{(-1)^m(2m-1)!!}{(1-2xt+t^2)^{m+1/2}} = \sum_{n=0}^{\infty}\mathcal{P}_n^m(x)t^{n-m} =$$

[set $s = n-m$
$n = s+m$
Lowest value of $s=0$, otherwise $\mathcal{P}_{s+m}^m = 0$]

$$= \sum_{s=0}^{\infty}\mathcal{P}_{s+m}^m t^s$$

Generating function $\boxed{\dfrac{(-1)^m(2m-1)!!}{(1-2xt+t^2)^{m+1/2}} \equiv g_m(x,t)}$

Take the derivative with respect to t:

$$\frac{\partial g_m}{\partial t} = \frac{(-m-\frac{1}{2})(-2x+2t)}{(1-2xt+t^2)^{m+3/2}} = \frac{(2m+1)(x-t)}{(1-2xt+t^2)} g_m(x,t)$$

so that $(1-2xt+t^2)\dfrac{\partial g_m}{\partial t} = (2m+1)(x-t)g_m(x,t)$

so that $(1-2xt+t^2)\sum_{s=0}^{\infty} \mathcal{P}_{s+m}^{m}\, s\, t^{s-1} = (2m+1)(x-t)\sum_{s=0}^{\infty}\mathcal{P}_{s+m}^{m} t^{s}$

$$\sum_{s=0}^{\infty}\mathcal{P}_{s+m}^{m} s t^{s-1} - 2x\sum_{s=0}^{\infty}\mathcal{P}_{s+m}^{m} s t^{s} + \sum_{s=0}^{\infty}\mathcal{P}_{s+m}^{m} s t^{s+1} =$$

$$= (2m+1)\sum_{s=0}^{\infty} x\,\mathcal{P}_{s+m}^{m} t^{s} - (2m+1)\sum_{s=0}^{\infty}\mathcal{P}_{s+m}^{m} t^{s+1}$$

collect equal powers:

t^s: $(s+1)\mathcal{P}_{s+m+1}^{m} - 2xs\,\mathcal{P}_{s+m}^{m} + \mathcal{P}_{s+m-1}^{m}\cdot(s-1) = (2m+1)x\,\mathcal{P}_{s+m}^{m} - (2m+1)\mathcal{P}_{s+m}^{m}$

$\Rightarrow (s+1)\mathcal{P}_{s+m+1}^{m} - (2m+1+2s)x\,\mathcal{P}_{s+m}^{m} + (s+2m)\mathcal{P}_{s+m-1}^{m} = 0$

set $l = s+m$, $s = l-m$

$$\Rightarrow \boxed{(l-m+1)\mathcal{P}_{l+1}^{m} - (2l+1)x\,\mathcal{P}_{l}^{m} + (l+m)\mathcal{P}_{l-1}^{m} = 0}$$

We have $g_{m+1}(x,t) = \dfrac{(-1)^{m+1}(2m+1)!!}{(1-2xt+t^2)^{m+3/2}} = -\dfrac{(2m+1)}{(1-2xt+t^2)} g_m(x,t)$

so that $(1-2xt+t^2)\,g_{m+1}(x,t) = -(2m+1)\,g_m(x,t)$

and $\sum_{s=0}^{\infty}\mathcal{P}_{s+m+1}^{m+1} t^{s} - 2x\sum_{s=0}^{\infty}\mathcal{P}_{s+m+1}^{m+1} t^{s+1} + \sum_{s=0}^{\infty}\mathcal{P}_{s+m+1}^{m+1} t^{s+2} = -(2m+1)\sum_{s=0}^{\infty}\mathcal{P}_{s+m}^{m} t^{s}$

collect coefficients for t^s:

$$\mathcal{P}_{s+m+1}^{m+1} - 2x\,\mathcal{P}_{s+m}^{m+1} + \mathcal{P}_{s+m-1}^{m+1} = -(2m+1)\mathcal{P}_{s+m}^{m}$$

set $s+m = l$: $\boxed{\mathcal{P}_{l+1}^{m+1} - 2x\,\mathcal{P}_{l}^{m+1} + \mathcal{P}_{l-1}^{m+1} = -(2m+1)\mathcal{P}_{l}^{m}}$

Since there are two indices to do recursion on a large number of recursion relations can be derived

e.g. $P_l^{m+1}(x) + \dfrac{2mx}{(1-x^2)^{1/2}} P_l^m(x) + (l+m)(l-m+1)P_l^{m-1}(x) = 0$ <u>recursion on m</u>

$(2l+1)x P_l^m(x) = (l+m)P_{l-1}^m(x) + (l-m+1)P_{l+1}^m(x)$ <u>recursion on l</u>

etc

The parity is found by investigating $P_\ell^m(-x)$

From the definition $P_\ell^m(-x) = (-1)^m(1-(-x)^2)^{m/2} \frac{d^m}{d(-x)^m} P_\ell(-x) =$

$$= (-1)^m(1-x^2)^{m/2}(-1)^m \frac{d^m}{dx^m}(-1)^\ell P_\ell(x) =$$

$$= (-1)^{\ell+m}(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x) = (-1)^{\ell+m}P_\ell^m(x)$$

$\therefore$ Parity given by $(-1)^{\ell+m}$

<u>Specific values</u>: $P_\ell^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$

For $m \neq 0$ $\quad P_\ell^m(\pm 1) = 0$

For $m = 0$ $\quad P_\ell^m \to P_\ell$ and $P_\ell(1) = 1$, $P_\ell(-1) = (-1)^\ell$

~~15.4.4a~~

$x = 0$: We have the recurrence relation

$$(2\ell+1)\,x\,P_\ell^m(x) = (\ell+m)P_{\ell-1}^m(x) + (\ell-m+1)P_{\ell+1}^m(x) = 0$$

which for $x=0$ becomes $\quad P_{\ell+1}^m(0) = -\frac{\ell+m}{\ell-m+1}P_{\ell-1}^m(0)$

15.4.4a) Specialize to $m=1$. Start with $P_1^1(0) = (-1)^1(1-x^2)^{1/2}\frac{d}{dx}\underbrace{x}_{P_1(x)}\Big|_{x=0}$

$$= -1$$

From $P_1^1(0) = -1$ we get $P_3^1(0) = -\frac{2+1}{2-1+1}(-1) = \frac{3}{2}$

$P_5^1(0) = -\frac{4+1}{4-1+1}\cdot\frac{3}{2} = -\frac{3\cdot 5}{4\cdot 2}$ and in general

$$P_{2\ell+1}^1(0) = (-1)^{(\ell+1)}\frac{(2\ell+1)!!}{(2\ell)!!}$$

$P_{2s}^1(0) = 0$ since the parity is odd $\left((-1)^{\ell+m}\right)$

15.4.5 General case from the generating function

$$g_m(0,t) = \frac{(-1)^m(2m-1)!!}{(1+t^2)^{m+1/2}} = (-1)^m(2m-1)!!\sum_{s=0}^{\infty}\binom{-m-\frac{1}{2}}{s}t^{2s} =$$

$$= \sum_{s=0}^{\infty}P_{2s+m}^m(0)\,t^{2s}$$

We obtain $P^m_{2s+m}(0) = (-1)^m (2m-1)!! \binom{-m-1/2}{s}$

Binomial coefficient $\binom{-m-1/2}{s} = \frac{1}{s!}\left(-\frac{2m+1}{2}\right)\left(-\frac{2m+3}{2}\right)\cdots\left(-\frac{2s+2m-1}{2}\right)$

$$= \frac{(-1)^s}{s!} \frac{(2m+2s-1)!!}{2^s (2m-1)!!}$$

and $P^m_{2s+m}(0) = (-1)^{m+s} \dfrac{(2m-1)!!\,(2s+2m-1)!!}{\underbrace{2^{2s}\, s!}_{(2s)!!}\,(2m-1)!!} = (-1)^{m+s} \dfrac{(2s+2m-1)!!}{(2s)!!}$

set $2s+m = \ell$, $s = \frac{1}{2}(\ell-m)$, $m+s = \frac{1}{2}(\ell+m)$

$$\Rightarrow P^m_\ell(0) = (-1)^{(\ell+m)/2} \frac{(\ell+m-1)!!}{(\ell-m)!!} \quad \text{for } \ell+m \text{ even}$$

and $P^m_\ell(0) = 0$ for $\ell+m$ odd

x ——————————————————————— x

Orthogonality and normalization

Since the associated Legendre functions are eigenfunctions of a Sturm-Liouville problem they are orthogonal. It can however easily be demonstrated using the Rodrigues formula

$$P^m_\ell(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

and the fact that $\frac{d^q}{dx^q}(x^2-1)^\ell = 0$ if $q > 2\ell$

To simplify write $R \equiv x^2-1$ and consider the integral

$$\int_{-1}^{1} P^m_p(x)\, P^m_q(x)\, dx = \frac{(-1)^{2m}}{2^{p+q} p!\, q!} \int_{-1}^{1} (-R)^m \left(\frac{d^{p+m}}{dx^{p+m}} R^p\right)\left(\frac{d^{q+m}}{dx^{q+m}} R^q\right) dx$$

$$= \frac{(-1)^m}{2^{p+q} p!\, q!} \int_{-1}^{1} R^m \left(\frac{d^{p+m}}{dx^{p+m}} R^p\right)\left(\frac{d^{q+m}}{dx^{q+m}} R^q\right) dx$$

Assume $p<q$ and integrate by parts $p+m+1$ times differentiating $u \equiv R^{m}\left(\frac{d^{p+m}}{dx^{p+m}} R^{p}\right)$ and integrating the same number of times $v \equiv \frac{d^{q+m}}{dx^{q+m}} R^{q}$

We have $p+m+1 \le q+m$ and the integrated terms $(u \cdot v)$ vanish at each step since there is always at least one factor R which gives zero for $x = \pm 1$.

We arrive at $$\frac{d^{p+m+1}}{dx^{p+m+1}} u = \frac{d^{p+m+1}}{dx^{p+m+1}} \Big(\underset{\sim x^{2m}}{R^{m}} \frac{d^{p+m}}{dx^{p+m}} \underset{\sim x^{2p}}{R^{p}} \Big) = 0$$

Note that $p \ge m$ and the highest degree : $R^{m} \frac{d^{p+m}}{dx^{p+m}} R^{p} \sim x^{2m} \cdot x^{2p-p-m} \sim x^{p+m}$

$\therefore$ Orthogonality $$\int_{-1}^{1} P_p^{m}(x)\, P_q^{m}(x)\, dx = 0 \quad \text{for } p \ne q$$

Consider $p=q$ ~~$(-1)^{p+m}$ from integrating by parts~~

The integral reads $$\int_{-1}^{1} [P_p^m(x)]^2 dx = \frac{(-1)^{\cancel{2}m\cancel{+p}}}{2^{2p}(p!)^2} \int_{-1}^{1} R^{m} \left(\frac{d^{p+m}}{dx^{p+m}} R^{p}\right)\left(\frac{d^{p+m}}{dx^{p+m}} R^{p}\right) dx$$

Integrate by parts $p+m$ times (maximum possible)
We still have the highest degree of $R^{m} \frac{d^{p+m}}{dx^{p+m}} R^{p}$ as x^{p+m} so we will get a constant contribution from this term.
The integrated parts still vanish

Leibniz's formula $$\frac{d^{p+m}}{dx^{p+m}}\left[R^{m} \frac{d^{p+m}}{dx^{p+m}} R^{p}\right] =$$
$$= \sum_{s=0}^{p+m} \binom{p+m}{s} \frac{d^{s}}{dx^{s}} R^{m} \frac{d^{2p+2m-s}}{dx^{2p+2m-s}} R^{p}$$

We have the constant term from $s = 2m$, all other terms vanish

Thus, $$\frac{d^{p+m}}{dx^{p+m}}\left[R^m \frac{d^{p+m}}{dx^{p+m}}R^p\right] = \binom{p+m}{2m}\left(\frac{d^{2m}}{dx^{2m}}R^m\right)\left(\frac{d^{2p}}{dx^{2p}}R^p\right)$$

$$= \frac{(p+m)!}{(2m)!(p-m)!}\cdot(2m)!\cdot(2p)! = \frac{(p+m)!}{(p-m)!}(2p)!$$

The normalization integral becomes

from integrating by parts $p+m$ times $(-1)^{p+m}$

$$\int_{-1}^{1}\left[P_p^m(x)\right]^2 dx = \frac{(-1)^{p+2m}}{2^{2p}\,p!\,p!}\,\frac{(p+m)!}{(p-m)!}(2p)!\int_{-1}^{1}R^p\,dx$$

The integral $$\int_{-1}^{1}R^p\,dx = \int_{-1}^{1}(x^2-1)^p\,dx = (-1)^p\int_{-1}^{1}(1-x^2)^p\,dx =$$

$$= \left[\begin{array}{l} x=\cos\vartheta \\ dx=-\sin\vartheta\,d\vartheta \\ x=-1\to\vartheta=\pi \\ x=1\to\vartheta=0 \end{array}\right] = (-1)^p\int_{\pi}^{0}\sin^{2p}\vartheta\,(-\sin\vartheta)\,d\vartheta =$$

$$= (-1)^p\int_{0}^{\pi}\sin^{2p+1}(\vartheta)\,d\vartheta = (-1)^p\,\frac{2^{2p+1}\,p!\,p!}{(2p+1)!}$$

Beta-function

x —— Beta function —— x

The gamma function $$\Gamma(p) = 2\int_0^\infty e^{-t^2}t^{2p-1}\,dt$$

Take $$\Gamma(p)\Gamma(q) = 4\int_0^\infty e^{-t^2}t^{2p-1}\,dt\int_0^\infty e^{-s^2}s^{2q-1}\,ds =$$

$$= 4\int_0^\infty\int_0^\infty e^{-(s^2+t^2)}\,t^{2p-1}s^{2q-1}\,ds\,dt = \left[\begin{array}{l} s=r\cos\vartheta \\ t=r\sin\vartheta \\ r^2=s^2+t^2 \\ ds\,dt=r\,dr\,d\vartheta \end{array}\right] =$$

$$= 4\int_0^\infty e^{-r^2}r^{2p+2q-2}\,r\,dr\int_0^{\pi/2}\cos^{2q-1}(\vartheta)\sin^{2p-1}(\vartheta)\,d\vartheta =$$

$$= 4\int_0^\infty e^{-r^2}r^{2p+2q-1}\,dr\int_0^{\pi/2}\cos^{2q-1}(\vartheta)\sin^{2p-1}(\vartheta)\,d\vartheta = 2\Gamma(p+q)\int_0^{\pi/2}\cos^{2q-1}(\vartheta)\sin^{2p-1}(\vartheta)$$

$$\therefore\; 2\int_0^{\pi/2}\cos^{2q-1}(\vartheta)\sin^{2p-1}(\vartheta)\,d\vartheta = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \equiv B(p,q)$$

Our integral $$\int_0^{\pi}\sin^{2p+1}(\vartheta)\,d\vartheta = 2\int_0^{\pi/2}\sin^{2p+1}(\vartheta)\,d\vartheta = B\left(p+1,\tfrac{1}{2}\right)$$

so that $\int_0^{\pi} \sin^{2p+1}(\vartheta)\, d\vartheta = \dfrac{\Gamma(p+1)\Gamma(\frac{1}{2})}{\Gamma(p+\frac{3}{2})} =$

$$= \frac{\sqrt{\pi}\; p!}{\sqrt{\pi}\,(2p+1)!!/2^{p+1}} = \frac{2\,(2^p p!)}{(2p+1)!!} = \frac{2^{p+1}\, p!}{(2p+1)!/2^p p!} =$$

$$= \frac{2^{2p+1}\, p!\, p!}{(2p+1)!}$$

Thus, $\displaystyle\int_{-1}^{1} \left[P_\ell^m(x)\right]^2 dx = \frac{(-1)^p}{2^{2p} p!\, p!}\, \frac{(p+m)!}{(p-m)!}\,(2p)!\cdot(-1)^p\,\frac{2^{2p+1} p!\, p!}{(2p+1)!} =$

$$= \frac{2}{2p+1}\cdot\frac{(p+m)!}{(p-m)!}$$

and $\boxed{\displaystyle\int_{-1}^{1} P_p^m(x)\, P_q^m(x)\, dx = \frac{2}{2p+1}\,\frac{(p+m)!}{(p-m)!}\,\delta_{pq}}$

Spherical harmonics

The solutions for the φ-dependence when separating the Laplace, Helmholtz or Schrödinger equations in spherical polar coordinates are $\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}\, e^{\pm i m\varphi}$, m integer

The $\Phi_m(\varphi)$ are orthonormal with $\int_0^{2\pi} [\Phi_m(\varphi)]^* \Phi_{m'}(\varphi)\, d\varphi = \delta_{mm'}$

For the associated Legendre functions we can define orthonormalized solutions to the ϑ-equation as

$$\Theta_{\ell m}(\vartheta) = \sqrt{\frac{2\ell+1}{2}\cdot\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\vartheta)$$

The product $\Theta_{\ell m}(\vartheta)\,\Phi_m(\varphi) \equiv Y_\ell^m(\vartheta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\cdot\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\vartheta)\, e^{im\varphi}$

is called spherical harmonic and are orthonormal on the unit sphere $\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta \left[Y_{\ell_1}^{m_1}(\vartheta,\varphi)\right]^* Y_{\ell_2}^{m_2}(\vartheta,\varphi) = \delta_{\ell_1\ell_2}\,\delta_{m_1 m_2}$

The spherical harmonics are eigenfunctions to the angular momentum operators in quantum mechanics with $\hat{L}^2 Y_\ell^m(\vartheta,\varphi) = \hbar^2\ell(\ell+1) Y_\ell^m(\vartheta,\varphi)$ and $\hat{L}_z Y_\ell^m(\vartheta,\varphi) = \hbar m Y_\ell^m(\vartheta,\varphi)$

Since the spherical harmonics are solutions to Sturm-Liouville problems they will form a complete basis ~~on~~ on the unit sphere and we can expand

$$f(\vartheta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell m} Y_\ell^m(\vartheta,\varphi)$$

where the coefficients are found through projection

$$c_{\ell m} = \langle Y_\ell^m | f(\vartheta,\varphi)\rangle = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, Y_\ell^m(\vartheta,\varphi)^* f(\vartheta,\varphi)\, d\vartheta$$

~~Exa~~ Properties: $Y_\ell^m(0,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m0}$ since $P_\ell^m(1) = 0$ if $m \neq 0$

similarly $Y_\ell^m(\pi,\varphi) = (-1)^\ell\sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m0}$

From the recurrence relations for $P_\ell^m(x)$ we have

$x \to \cos\vartheta$:
$$\cos\vartheta\, Y_\ell^m = \left[\frac{(\ell-m+1)(\ell+m+1)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y_{\ell+1}^m + \left[\frac{(\ell-m)(\ell+m)}{(2\ell-1)(2\ell+1)}\right]^{1/2} Y_{\ell-1}^m$$

and
$$e^{\pm i\varphi}\sin\vartheta\, Y_\ell^m = \mp\left[\frac{(\ell\pm m+1)(\ell\pm m+2)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y_{\ell+1}^{m\pm1} \pm \left[\frac{(\ell\mp m)(\ell\mp m\mp 1)}{(2\ell-1)(2\ell+1)}\right]^{1/2} Y_{\ell-1}^{m\pm1}$$

These are important for example when establishing selection rules in absorption or emission of radiation in a dipole process with ~~...~~ initial and final states Y_ℓ^m and $Y_{\ell'}^{m'}$ through

$$\int [Y_{\ell'}^{m'}]^* \cos\vartheta\, Y_\ell^m\, d\Omega =$$

$$= \left[\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}\right]^{1/2} \delta_{m'm}\,\delta_{l',l+1} + \left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2} \delta_{m'm}\,\delta_{l,l-1}$$

× ———— The second kind... ———— ×

The Legendre equation as a linear second-order ODE has two linearly independent solutions

Write $y'' - \frac{2x}{1-x^2} y' - \frac{l(l+1)}{1-x^2} y = 0$ (standard form for second solution)

with $l \geq 0$.

From the known solutions $P_l(x)$ we can generate solutions

$$Q_l(x) = P_l(x) \int^x \frac{\exp\left[\int^s \frac{2t}{1-t^2}\,dt\right]}{[P_l(s)]^2}\,ds =$$

$$= P_l(x) \int^x \frac{ds}{(1-s^2)[P_l(s)]^2}$$

Generate $Q_0(x)$ and $Q_1(x)$ and then proceed using recurrence relations

$$Q_0(x) = \int^x \frac{ds}{1-s^2} = \frac{1}{2}\int \frac{ds}{1-x} + \frac{1}{2}\int \frac{ds}{1+x} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

partial fraction $\frac{1}{(1-s)(1+s)} = \frac{a}{1-s} + \frac{b}{1+s} = \frac{a + as + b - bs}{(1-s^2)}$ $\left.\begin{array}{l} a+b=1 \\ a-b=0 \end{array}\right\} a=b=\frac{1}{2}$

$$Q_1(x) = x \int_{\cancel{0}}^x \frac{ds}{(1-x^2)x^2} = x \int_{\cancel{0}}^x \left\{ \frac{1}{2}\left(\frac{1}{1-s} + \frac{1}{1+s}\right) + \frac{1}{s^2} \right\} ds =$$

$$= \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

Since $Q_l(x)$ satisfy the Legendre equation they obey the same recurrence relations and

$$(l+1)\, Q_{l+1}(x) - (2l+1)\, x\, Q_l(x) + l\, Q_{l-1}(x) = 0$$

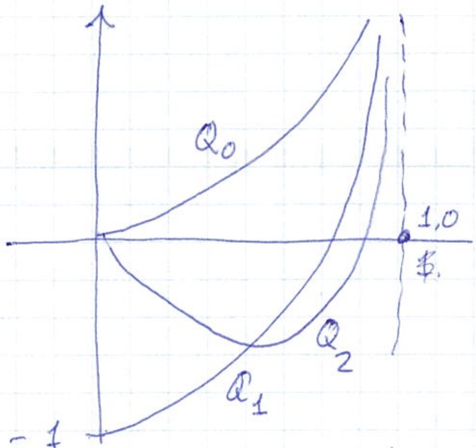
$$(2l+1)\, Q_l(x) = Q'_{l+1}(x) - Q'_{l-1}(x)$$

Range of definition $-1 < x < +1$ can be extended to $x > 1$ which gives

$$\ln\left(\frac{1+x}{1-x}\right) = \ln\left(-\frac{x+1}{x-1}\right) = \ln\left(e^{i\pi}\,\frac{x+1}{x-1}\right) =$$

$$= i\pi + \ln\left(\frac{x+1}{x-1}\right)$$ which gives $i\pi\, P_l(x)$ as contribution;

logarithmic singularity at $x = 1$ (regular at $x=0$)



$0 \leq x < 1$