Hankel functions

The Bessel functions $J_\nu(x)$ and Neumann functions $Y_\nu(x)$ have asymptotic properties suitable to describe standing waves. For problems involving traveling waves in spherical or cylindrical symmetry the Hankel functions are useful

$$H_\nu^{(1)}(x) = J_\nu(x) + i Y_\nu(x)$$
$$H_\nu^{(2)}(x) = J_\nu(x) - i Y_\nu(x) \qquad \text{cf. } e^{\pm i\vartheta} = \cos\vartheta \pm i\sin\vartheta$$

The series expansions are obtained by combining the expansions for $J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(\nu+s+1)} \left(\frac{x}{2}\right)^{\nu+2s}$ and the corresponding for $Y_\nu(x)$ where for the first terms it suffices to know

$$Y_\nu(x) \approx -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu + \dots$$

and $Y_0(x) = \frac{2}{\pi} J_0(x)\left[\gamma + \ln\left(\frac{x}{2}\right)\right] - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,k!} H_k \left(\frac{x}{2}\right)^{2k}$

where γ is the Euler-Mascheroni constant and

$$H_m = \sum_{j=1}^{m} \frac{1}{j}$$

small x

Then $H_0^{(1)}(x) \approx \underset{\text{from } J_0(x)}{1} + \underbrace{\frac{2i}{2\pi}\ln x + \frac{2i}{2\pi}[\gamma - \ln 2]}_{\text{from } J_0(x)[\gamma + \ln\frac{x}{2}]} + \dots$

$$H_\nu^{(1)}(x) \approx -i\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu + \dots, \quad \nu > 0$$

$$H_0^{(2)}(x) = \left(H_0^{(1)}(x)\right)^* \approx -\frac{2i}{2\pi}\ln x + 1 - \frac{2i}{2\pi}[\gamma - \ln 2] + \dots \quad (x \text{ real})$$
$$H_\nu^{(2)}(x) = \left(H_\nu^{(1)}(x)\right)^* \approx \frac{i}{\pi}\Gamma(\nu)\left(\frac{2}{x}\right)^\nu + \dots, \quad \nu > 0 \quad (x \text{ real})$$

- Recurrence relations: The Hankel functions obey the same recurrence relations as their constituents J_ν, Y_ν

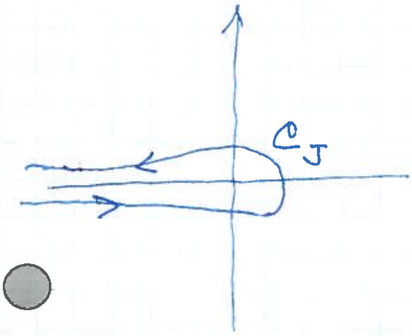
i.e. $H_{\nu-1}(x) + H_{\nu+1}(x) = \frac{2\nu}{x} H_\nu(x)$

& $H_{\nu-1}(x) - H_{\nu+1}(x) = 2H_\nu'(x)$

Contour integral representation of the Hankel functions:

For $J_\nu(x)$ we have the integral representation

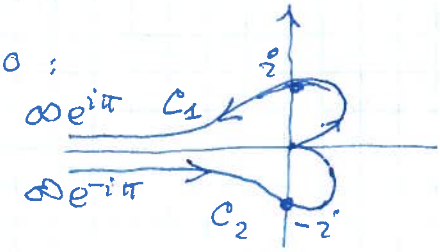
$$J_\nu(x) = \frac{1}{2\pi i}\int_C e^{(x/2)(t-\frac{1}{t})}\frac{dt}{t^{\nu+1}} \quad \text{with contour } C$$



We found that Bessel's ODE was satisfied for any open contour for which an expression of the form $\frac{e^{(x/2)(t-\frac{1}{t})}}{t^\nu}\left[\nu + \frac{x}{2}\left(t+\frac{1}{t}\right)\right]$ vanished at both endpoints.

Deform the contour to:

saddle points at $t = \pm i$



The expression vanishes also at $t = 0^+$ due to $e^{(x/2)(t-\frac{1}{t})}$ as long as the approach is from positive t

Both contours C_1 and C_2 give solutions to the Bessel ODE since they both have vanishing integrand of the equation at the endpoints.

Set $H_\nu^{(1)}(x) = \frac{1}{\pi i}\int_{C_1} e^{(x/2)(t-\frac{1}{t})}\frac{dt}{t^{\nu+1}}$

and $H_\nu^{(2)}(x) = \frac{1}{\pi i}\int_{C_2} e^{(x/2)(t-\frac{1}{t})}\frac{dt}{t^{\nu+1}}$

Combining the paths we have a contour giving $J_\nu(x)$

so $J_\nu(x) = \frac{1}{2}\left[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)\right]$

We now need to show that $Y_\nu(x) = \frac{1}{2i}\left[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)\right]$ from the integral expressions corresponding to the definitions of $H_\nu^{(i)}(x)$

In order to construct $Y_\nu(x)$ from the integral representations we need $J_{-\nu}(x)$ in terms of the contour integrals giving the Hankel functions.

In $H_\nu^{(1)}(x) = \frac{1}{\pi i}\int_{C_1} e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} \frac{dt}{t^{\nu+1}}$ make the substitution $t = \frac{e^{i\pi}}{s}$

$s = \frac{e^{i\pi}}{t}$ maps $t = 0+$ to $s = e^{i\pi}\infty$

$t = i \rightarrow s = \frac{e^{i\pi}}{i} = i$

$t = e^{i\pi}\infty \rightarrow s = 0+$

$\therefore$ Same contour but traverse in the opposite sense (minus sign introduced)

So $H_\nu^{(1)}(x) = -\frac{1}{\pi i}\int_{C_1} e^{\frac{x}{2}\left(s-\frac{1}{s}\right)} \frac{ds/s^2}{\left(\frac{e^{i\pi}}{s}\right)^{\nu+1}} =$

$$= \frac{e^{-i\nu\pi}}{\pi i}\int_{C_1} e^{\left(\frac{x}{2}\right)\left(s-\frac{1}{s}\right)} \frac{ds}{s^{-\nu+1}} = e^{-i\nu\pi} H_{-\nu}^{(1)}(x)$$

For the contour C_2 use the transformation $t = \frac{e^{-i\pi}}{s}$ to reach $e^{-i\pi}\infty$. Also this transformed contour is the same as the original C_2 but transversed in the opposite sense

We get $H_\nu^{(2)}(x) = e^{i\nu\pi} H_{-\nu}^{(2)}(x)$

So $J_{-\nu}(x) = \frac{1}{2}\left[H_{-\nu}^{(1)}(x) + H_{-\nu}^{(2)}(x)\right] = \frac{1}{2}\left[e^{i\nu\pi} H_\nu^{(1)}(x) + e^{-i\nu\pi} H_\nu^{(2)}(x)\right]$

And $Y_\nu(x) = \frac{\cos\nu\pi\, J_\nu - J_{-\nu}}{\sin\nu\pi} = \frac{\frac{1}{2}\left(e^{i\nu\pi}+e^{-i\nu\pi}\right)\frac{1}{2}\left(H_\nu^{(1)}+H_\nu^{(2)}\right) - \frac{1}{2}\left(e^{i\nu\pi}H_\nu^{(1)} + e^{-i\nu\pi}H_\nu^{(2)}\right)}{\sin\nu\pi}$

$$= \frac{\frac{1}{4}\left(e^{i\nu\pi}H_\nu^{(1)} + e^{-i\nu\pi}H_\nu^{(1)} + e^{i\nu\pi}H_\nu^{(2)} + e^{-i\nu\pi}H_\nu^{(2)} - 2e^{i\nu\pi}H_\nu^{(1)} - 2e^{-i\nu\pi}H_\nu^{(2)}\right)}{\sin\nu\pi} =$$

$$= \frac{1}{\sin\nu\pi}\cdot\frac{1}{4}\left(-H_\nu^{(1)}\underbrace{\left(e^{i\nu\pi} - e^{-i\nu\pi}\right)}_{2i\sin\nu\pi} + H_\nu^{(2)}\underbrace{\left(e^{i\nu\pi} - e^{-i\nu\pi}\right)}_{2i\sin\nu\pi}\right) =$$

$$= \frac{i}{2}\left(-H_\nu^{(1)} + H_\nu^{(2)}\right) = \frac{1}{2i}\left(H_\nu^{(1)} - H_\nu^{(2)}\right)$$

Modified Bessel functions $I_\nu(x)$, $K_\nu(x)$

Depending on the separation constant the sign may be opposite to what is required to yield Bessel's DE.

$$\rho^2 \frac{d^2}{d\rho^2} P_\nu(k\rho) + \rho \frac{d}{d\rho} P_\nu(k\rho) - (k^2\rho^2 + \nu^2) P_\nu(k\rho) = 0$$

Solutions are modified Bessel functions and are not oscillatory (cf $\{\sin x, \cos x\}$ vs $\{\sinh x, \cosh x\}$)

The substitution $k \to ik$ gives the Bessel equation so the solutions are

$$J_\nu(ix) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s+\nu+1)} \left(\frac{ix}{2}\right)^{\nu+2s} = i^\nu \sum_{s=0}^{\infty} \frac{1}{s!\,\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2s}$$

Define $I_\nu(x) = i^{-\nu} J_\nu(ix) = e^{-i\nu\pi/2} J_\nu(x e^{i\pi/2}) \Rightarrow I_\nu(x)$ real for real x

Asymptotically for small x $\quad I_\nu(x) \approx \dfrac{x^\nu}{2^\nu \Gamma(\nu+1)}$ for $\nu \geq 0$

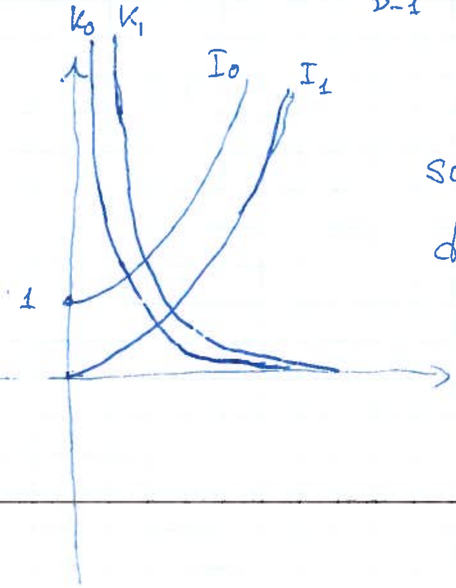
For integer ν: $\quad I_{-n}(x) = i^n J_{-n}(ix) = i^n (-1)^n J_n(ix) = i^n (-1)^n (i^n I_n(x))$

$$= i^{2n}(-1)^n I_n(x) = I_n(x)$$

Recurrence relations for I_ν derived from those of J_ν by substituting $x \to ix$ and $J_\nu(ix) = i^\nu I_\nu(x)$

e.g. $\quad I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x) \quad$ (cf $J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu$)

$\quad I_{\nu-1}(x) + I_{\nu+1}(x) = 2 I'_\nu(x) \quad$ (cf. $J_{\nu-1} - J_{\nu+1} = 2J'_\nu$)



If ν is an integer we only have one independent solution and we need a second. This can be defined in various ways. Here we define it in terms of the Hankel function $H^{(1)}_\nu(x)$ as

$$K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1} H^{(1)}_\nu(ix) = \frac{\pi}{2} i^{\nu+1} \left[J_\nu(ix) + i Y_\nu(ix) \right]$$

$K_\nu(x)$ like this is real when x is real.

Using $Y_\nu(x) = \dfrac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}$ and $J_\nu(xi) = i^{\nu} I_\nu(x)$

we have $\dfrac{2}{\pi}K_\nu(x) = i^{\nu+1}\{J_\nu(ix) + i\,Y_\nu(ix)\} =$

$$= i^{\nu+1}\left\{ i^{\nu} I_\nu(x) + i\,\frac{i^{\nu}\cos\nu\pi\, I_\nu(x) - i^{-\nu}I_{-\nu}(x)}{\sin\nu\pi}\right\} =$$

$$= i\left\{\frac{i^{2\nu}(\sin\nu\pi + i\cos\nu\pi)I_\nu - i I_{-\nu}}{\sin\nu\pi}\right\} =$$

$$= \frac{I_{-\nu} - i^{2\nu}(\cos\nu\pi - i\sin\nu\pi)I_\nu}{\sin\nu\pi} = \frac{I_{-\nu} - e^{\frac{i\pi}{2}\cdot 2\nu}\, e^{-\pi\nu}\, I_\nu}{\sin\nu\pi} =$$

$$= \frac{I_{-\nu} - I_\nu}{\sin\nu\pi} \qquad \therefore\; K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$$ limit implied for $\nu = n$

The integral representation of $I_0(x)$ can be obtained from

$$I_0(x) = J_0(ix) = \frac{1}{\pi}\int_0^{\pi}\cos(ix\sin\vartheta)\,d\vartheta = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\left(e^{-x\sin\vartheta} + e^{x\sin\vartheta}\right)d\vartheta =$$

$$= \frac{1}{\pi}\int_0^{\pi}\cosh(x\sin\vartheta)\,d\vartheta$$

for $K_0(x)$ we have $K_0(x) = \displaystyle\int_0^{\infty}\cos(x\sinh t)\,dt = \int_0^{\infty}\frac{\cos xt}{(t^2+1)^{1/2}}\,dt$ $\quad x>0$

1-D Green's function for the Laplace equation in cylindrical coord.

Equation $\left\{\dfrac{\partial^2}{\partial\rho_1^2} + \dfrac{1}{\rho_1}\dfrac{\partial}{\partial\rho_1} - \dfrac{m^2}{\rho_1^2} - k^2\right\} g_m(k,\rho_1,\rho_2) = \delta(\rho_1 - \rho_2)$

Modified Bessel equation with solutions $I_m(k\rho)$ and $K_m(k\rho)$

I_m is regular at the origin while K_m is regular at infinity

Write $g_m(k,\rho_1,\rho_2) = A_m I_m(k\rho_<)K_m(k\rho_>)$ where $\rho_<$ is the lesser of ρ_1 and ρ_2 and $\rho_>$ is the greater of ρ_1 and ρ_2. The coefficient A_m is determined from the Wronskian $k\rho\left(K_m'(k\rho)I_m(k\rho) - I_m'(k\rho)K_m(k\rho)\right)^{-1}$

Write $K_m(x) = \dfrac{\pi}{2} i^{m+1} H_m^{(1)}(ix)$, $I_m(x) = i^{-m}J_m(x)$

where $x = k\rho$

Thus $I_m(x)K_m'(x) - I_m'(x)K_m = i^{-m}J_m(ix)\left(\frac{\pi}{2}i^{m+1}\cdot i H^{(1)\prime}(ix)\right) -$

$$- i^{-m+1}J_m'(ix)\frac{\pi}{2}i^{m+1}H_m^{(1)}(ix) =$$

$$= \frac{\pi}{2}\left\{J_m'(ix)H_m^{(1)}(ix) - J_m(ix)H^{(1)\prime}(ix)\right\} =$$

$$= -\frac{\pi}{2}\left\{J_m(ix)H_m^{(1)\prime}(ix) - J_m'(ix)H_m^{(1)}(ix)\right\} =$$

$$= -\frac{\pi}{2}\left\{J_m(ix)\{J_m'(ix)+iY_m'(ix)\} - J_m'(ix)\{J_m(ix)+iY_m(ix)\}\right\} =$$

$$= -\frac{\pi}{2}i\left\{J_m(ix)Y_m'(ix) - J_m'(ix)Y_m(ix)\right\} = -\frac{\pi}{2}i\cdot\frac{2}{\pi i x} = -\frac{1}{x}$$

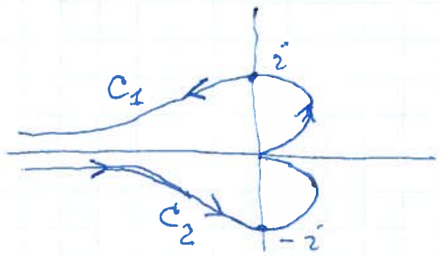
So $g_m(k,\rho_1,\rho_2) = -I_m(k\rho_<)K_m(k\rho_>)$

Asymptotic forms of Hankel functions

Take $H_\nu^{(1)}(t)$ and $H_\nu^{(2)}(t)$ as defined by the contour integrals

$$H_\nu^{(1)}(t) = \frac{1}{\pi i}\int_{C_1} e^{\left(\frac{t}{2}\right)\left(z-\frac{1}{z}\right)}\frac{dz}{z^{\nu+1}}$$

$$H_\nu^{(2)}(t) = \frac{1}{\pi i}\int_{C_2} e^{\left(\frac{t}{2}\right)\left(z-\frac{1}{z}\right)}\frac{dz}{z^{\nu+1}}$$



Regard the integrand as containing a slowly varying function $g(z,t) = z^{-\nu-1}$ and an exponential e^w with $w = \frac{t}{2}\left(z-\frac{1}{z}\right)$

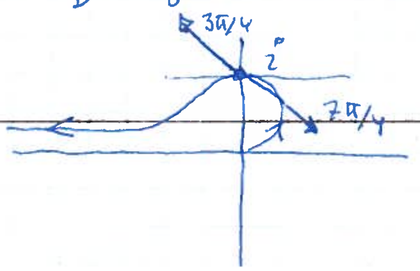
$$\int_C g(z,t)e^{w(z,t)}dz \approx g(z_0,t)e^{w(z_0,t)}e^{i\vartheta}\sqrt{\frac{2\pi}{|w''(z_0,t)|}}$$

where C passes through a saddle point at z_0 and

$$\vartheta = -\frac{\arg(w''(z_0,t))}{2} + \left(\frac{\pi}{2} \text{ or } \frac{3\pi}{2}\right) \quad \text{(Method of Steepest Descents)}$$

Find the zeros of $w' = \frac{t}{2}\left(1+\frac{1}{z^2}\right) \rightarrow z_0 = \pm i$

For $H_\nu^{(1)}$ $z_0 = i$: $w(i) = it$ $\quad w''(i) = -\frac{t}{z_0^3}\Big|_{z_0=i} = -it \quad \arg(w''(i)) = -\frac{\pi}{2}$



$\vartheta = \frac{3\pi}{4}$ which gives

$$H_\nu^{(1)}(t) \approx \frac{1}{\pi i}\underbrace{e^{\left(i\frac{\pi}{2}\right)(-\nu-1)}}_{i^{-\nu-1}}\cdot e^{it}e^{3i\pi/4}\sqrt{\frac{2\pi}{t}}$$

$\therefore \quad H_\nu^{(1)}(t) \approx \sqrt{\frac{2}{\pi t}}\, e^{i(t - \frac{\nu\pi}{2} - \frac{\pi}{4})}$

Similarly $\quad H_\nu^{(2)}(t) \approx \sqrt{\frac{2}{\pi t}}\, e^{-i(t - \nu\pi/2 - \frac{\pi}{4})}$

} oscillatory traveling waves

Since $\quad K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \approx \sqrt{\frac{2}{\pi i x}} \cdot \frac{\pi}{2} i^{\nu+1} \cdot e^{-x} \cdot e^{-i(\nu\frac{\pi}{2} + \frac{\pi}{4})}$

$\sim \sqrt{\frac{\pi}{2x}}\, e^{-x}$ exponentially decaying, irregular at the origin

and $\quad J_\nu(x) = \frac{1}{2}\left(H_\nu^{(1)}(x) + H_\nu^{(2)}(x)\right)$

$Y_\nu(x) = \frac{1}{2i}\left(H_\nu^{(1)}(x) - H_\nu^{(2)}(x)\right)$

$I_\nu(x)$ will be finite at the origin ($\nu \geq 0$) and increase exponentially at large x

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Spherical Bessel functions

Radial equation resulting from separating the Helmholtz equation in spherical coordinates

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [k^2r^2 - \ell(\ell+1)]R = 0$$

Substitute $\quad R(kr) = \frac{Z(kr)}{(kr)^{1/2}} \equiv \frac{Z(\rho)}{\rho^{1/2}} \quad (\rho = kr)$

$$\frac{dR}{d\rho} = \frac{Z'}{\rho^{1/2}} - \frac{1}{2}\frac{Z}{\rho^{3/2}} \quad ; \quad \frac{d^2R}{d\rho^2} = \frac{Z''}{\rho^{1/2}} - \frac{Z'}{\rho^{3/2}} + \frac{3}{4}\frac{Z}{\rho^{5/2}}$$

$$\Rightarrow \rho^2\frac{Z''}{\rho^{1/2}} - \frac{\rho Z'}{\rho^{1/2}} + \frac{3}{4}\frac{Z}{\rho^{1/2}} + 2\rho\frac{Z'}{\rho^{1/2}} - \frac{Z}{\rho^{1/2}} + [\rho^2 - \ell(\ell+1)]\frac{Z}{\rho^{1/2}} = 0$$

$$\therefore \rho^2 Z'' + \rho Z' + [\rho^2 \underbrace{- \ell(\ell+1) - \tfrac{1}{4}}_{(\ell+\frac{1}{2})^2}] Z = 0$$

$$\Rightarrow \rho^2 Z'' + \rho Z' + [\rho^2 - (\ell + \tfrac{1}{2})^2] Z = 0 \qquad \text{Bessel D.E}$$

$\therefore$ Z is a Bessel function with index $\ell + \frac{1}{2}$

Definitions (since the DE is homogeneous the scaling is for convenience)

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x) \qquad h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}}\, H^{(1)}_{n+\frac{1}{2}}(x) = j_n(x) + i y_n(x)$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{n+\frac{1}{2}}(x) \qquad h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}}\, H^{(2)}_{n+1/2}(x) = j_n(x) - i y_n(x)$$

We have a relation $y_n(x) = (1)^{n+1} j_{-n-1}(x)$

from $$Y_{n+\frac{1}{2}}(x) = \frac{\cos[(n+\frac{1}{2})\pi]\, J_{n+\frac{1}{2}}(x) - J_{-n-1/2}(x)}{\sin[(n+\frac{1}{2})\pi]} = (-1)^{n+1} J_{-n-\frac{1}{2}}(x)$$

Develop series forms from those of $J_{n+\frac{1}{2}}$ and $Y_{n+\frac{1}{2}}$

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x) = \sqrt{\frac{\pi}{2x}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{2s+n+\frac{1}{2}}$$

write $\Gamma(s+n+\frac{3}{2}) = \Gamma(n+\frac{3}{2})(n+\frac{3}{2})_s$

$(n+\frac{3}{2})_s = (n+\frac{3}{2})(n+\frac{5}{2})\dots(n+\frac{3}{2}+s-1)$ is a Pochhammer symbol

$(a)_0 = 1,\ (a)_1 = a,\ (a)_{n+1} = a(a+1)\dots(a+n)$

e.g. $$\binom{m}{n} = \frac{m!}{n!(m-n)!} = \frac{(m-n+1)_n}{n!} \leftarrow \text{gives the remainder}$$

so $$j_n(x) = \sqrt{\frac{\pi}{2x}} \left(\frac{x}{2}\right)^{n+\frac{1}{2}} \frac{1}{\Gamma(n+\frac{3}{2})} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+\frac{3}{2})_s} \left(\frac{x}{2}\right)^{2s} =$$

$$= \frac{x^n}{(2n+1)!!} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+\frac{3}{2})_s} \left(\frac{x}{2}\right)^{2s}$$ regular at the origin

where we used $\Gamma(n+\frac{3}{2}) = (n+\frac{1}{2})\Gamma(n+\frac{1}{2}) = \frac{2n+1}{2} \frac{(2n-1)!!}{2^n}\sqrt{\pi} =$

$$= \frac{(2n+1)!!}{2^{n+1}}\sqrt{\pi}$$

similarly $$y_n(x) = -\frac{(2n-1)!!}{x^{n+1}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(\frac{1}{2}-n)_s} \left(\frac{x}{2}\right)^{2s}$$ irregular at the origin

Performing the sum for $j_0(x)$ we find

$$j_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \, 2^{2s}}{(2s)!!\,(2s+1)!!} \left(\frac{x}{2}\right)^{2s} = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \, x^{2s} = \frac{\sin x}{x}$$

$s! = 2^{-s}(2s)!!$

$\left(\frac{3}{2}\right)_s = 2^{-s}(2s+1)!!$

$$y_0(x) = -\frac{\cos x}{x}$$

$$h_0^{(1)}(x) = j_0(x) + i\, y_0(x) = \frac{1}{x}(\sin x - i\cos x) = -\frac{i}{x} e^{ix}$$

$$h_0^{(2)}(x) = j_0(x) - i\, y_0(x) = \frac{i}{x} e^{-ix}$$

From recurrence relations all j_n and y_n will be linear combinations of sines and cosines

Recurrence relations: substitute $n+\frac{1}{2}$ into the recurrence relations for the integer functions or use the series form

$$f_{n-1}(x) + f_{n+1}(x) = \frac{2n+1}{x} f_n(x) \qquad f_n \in j_n, y_n, h_n^{(1)}, h_n^{(2)}$$

$$n f_{n-1}(x) - (n+1) f_{n+1}(x) = (2n+1) f_n'(x)$$

$$\frac{d}{dx}\left[x^{n+1} f_n(x)\right] = x^{n+1} f_{n-1}(x)$$

$$\frac{d}{dx}\left[x^{-n} f_n(x)\right] = -x^{-n} f_{n+1}(x)$$

For small x: $j_n(x) \approx \dfrac{x^n}{(2n+1)!!}$ regular $\qquad y_n(x) \approx -\dfrac{(2n-1)!!}{x^{n+1}}$ irregular

$x \gg 1$: $j_n(x) \sim \frac{1}{x}\sin\left(x - \frac{n\pi}{2}\right)$; $y_n(x) \sim -\frac{1}{x}\cos\left(x - \frac{n\pi}{2}\right)$

$$\left.\begin{aligned} h_n^{(1)}(x) &\sim -i\,\frac{e^{i(x-n\pi/2)}}{x} \\ h_n^{(2)}(x) &\sim i\,\frac{e^{-i(x-n\pi/2)}}{x} \end{aligned}\right\} \text{traveling waves}$$

Orthogonality and normalization

$$\int_0^a j_n\left(\alpha_{np}\frac{r}{a}\right) j_n\left(\alpha_{nq}\frac{r}{a}\right) r^2 dr = \frac{a^3}{2}\left[j_{n+1}(\alpha_{np})\right]^2 \delta_{pq}$$

where α_{ni} are zeros of j_n placed at $r=a$

Modified spherical Bessel functions

Radial equation $r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \left[k^2r^2 + \ell(\ell+1)\right]R = 0$

gives spherical Bessel functions with imaginary argument

define $\quad i_n(x) = \sqrt{\frac{\pi}{2x}}\, I_{n+\frac{1}{2}}(x)$

$$k_n(x) = \sqrt{\frac{2}{\pi x}}\, K_{n+1/2}(x)$$

Functional forms

$$i_0(x) = \frac{\sinh x}{x} \qquad k_0(x) = \frac{e^{-x}}{x}$$

$$i_1(x) = \frac{\cosh x}{x} - \frac{\sinh x}{x^2} \qquad k_1(x) = e^{-x}\left(\frac{1}{x} + \frac{1}{x^2}\right)$$

$\vdots$

Note:

eq. (14.199) $A\, j_\ell'(ka) = B\, k_\ell(k'a)$