Chapter 14

The Bessel functions are obtained as solutions for example to the wave equation in cylindrical coordinates and also in spherical polar coordinates. The solutions to the Schrödinger equation for a particle in a spherically symmetric box with impenetrable walls are spherical Bessel functions where the boundary condition $\psi(r=a)=0$ is satisfied by $J_\nu(\alpha_{m\nu})=0$, i.e. $\alpha_{m\nu}$ is a root to $J_\nu(x)=0$

We start from the generating function $g(x,t)=e^{\frac{x}{2}(t-\frac{1}{t})}$ which we expand in a Laurent series in t

$$e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty} J_n(x)t^n$$

The coefficients $J_n(x)$ are Bessel functions of the first kind with integer index. To find these functions we consider the generating function as a product $e^{xt/2}\cdot e^{-\frac{x}{2t}}$ and expand as a product of two MacLaurin expansions

$$e^{xt/2}\cdot e^{-\frac{x}{2t}}=\sum_{r=0}^{\infty}\left(\frac{x}{2}\right)^r\frac{t^r}{r!}\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^s\frac{t^{-s}}{s!}=$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{r+s}\frac{t^{r-s}}{r!\,s!}$$

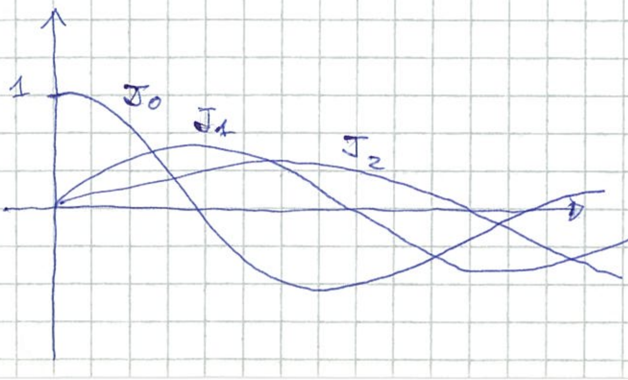
Let $n=r-s$ $\quad -s\le n\le\infty$, $\quad r=n+s$

$$=\sum_{n=-s}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s}{(n+s)!\,s!}\left(\frac{x}{2}\right)^{n+2s}t^n=\sum_{s=0}^{\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^s}{(n+s)!\,s!}\left(\frac{x}{2}\right)^{n+2s}t^n$$

So that, given n, we have

$$J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{(n+s)!\,s!}\left(\frac{x}{2}\right)^{n+2s}=$$

$$=\frac{x^n}{2^n n!}-\frac{x^{n+2}}{2^{n+2}(n+1)!}+\cdots$$



Generalize to nonintegers

$$J_\nu(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,\Gamma(\nu+s+1)}\left(\frac{x}{2}\right)^{\nu+2s}$$

$\nu\neq -1,-2,-3,\ldots$

Note: For negative n $\quad (s-|n|)!\to\infty$ for $|n|>s$

For n negative we get $J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s-n)!}\left(\frac{x}{2}\right)^{2s-n}$

n is an integer such that for $s<n$ $(s-n)! \to \infty$ and those terms do not contribute. Transform $s \to s+n$

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{s!(s+n)!}\left(\frac{x}{2}\right)^{n+2s} = (-1)^n J_n(x) \quad (n \text{ integer})$$

From the generating function we can obtain recursion relations through differentiation and from these we can construct the differential equation that $J_n(x)$ solve.

1) Differentiate $g(x,t)$ with respect to t:

$$\frac{\partial}{\partial t} g(x,t) = \frac{\partial}{\partial t} e^{\frac{x}{2}(t-\frac{1}{t})} = \frac{1}{2}x\left(1+\frac{1}{t^2}\right)e^{\frac{x}{2}(t-\frac{1}{t})}$$

Differentiate the expansion $\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty} J_n(x)t^n = \sum_{n=-\infty}^{\infty} n J_n(x) t^{n-1}$

We obtain $\frac{1}{2}x\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)t^n = \sum_{n=-\infty}^{\infty} n J_n(x) t^{n-1}$

$$\sum_{n=-\infty}^{\infty}\left\{J_n(x)+J_{n+2}(x)\right\}t^n = \frac{2}{x}\sum_{n=-\infty}^{\infty}(n+1)J_{n+1}(x)t^n$$

The coefficients of each power of t must be equal which gives

$$J_n(x) + J_{n+2}(x) = \frac{2}{x}(n+1)J_{n+1}(x)$$

$$n \to n-1 \Rightarrow J_{n+1}(x) = \frac{2}{x} n J_n(x) - J_{n-1}(x)$$

i.e. from J_{n-1} and J_n we obtain J_{n+1}

2) Differentiate with respect to x:

$$\frac{\partial}{\partial x} g(x,t) = \frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{x}{2}(t-\frac{1}{t})} = \frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{n=-\infty}^{\infty} J_n(x)t^n$$

The expansion: $\frac{\partial}{\partial x}\sum_{n=-\infty}^{\infty} J_n(x)t^n = \sum_{n=-\infty}^{\infty} J_n'(x)t^n$

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}\left\{J_{n-1}(x) - J_{n+1}(x)\right\}t^n = \sum_{n=-\infty}^{\infty} J_n'(x)t^n$$

$$\Rightarrow J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

Combining the recursion relations we obtain a one-step relation:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

$$\oplus \quad J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x)$$

$$2 J_{n-1}(x) = \frac{2n}{x} J_n(x) + 2 J_n'(x)$$

Divide by 2 and multiply by x^n

$$x^n J_{n-1}(x) = n x^{n-1} J_n(x) + x^n J_n' = \frac{d}{dx}\left[x^n J_n\right]$$

Similarly for J_{n+1} after subtraction

$$2 J_{n+1} = \frac{2n}{x} J_n - 2 J_n'$$. Divide by 2 and multiply by x^{-n}

$$x^{-n} J_{n+1} = n x^{-n-1} J_n - x^{-n} J_n' = -\frac{d}{dx}\left[x^{-n} J_n\right]$$

Bessel D.E.

Let the functions Z_ν satisfy the recursion relations where now ν do not need to be integers and Z_ν are not necessarily given by the series corresponding to $J_n(x)$

For the derivative we have

(1) $x Z_\nu'(x) = x Z_{\nu-1}(x) - \nu Z_\nu(x)$

Differentiate: $x Z_\nu'' + Z_\nu' - x Z_{\nu-1}' - Z_{\nu-1} + \nu Z_\nu' = 0$

$$\Rightarrow x Z_\nu'' + (\nu+1) Z_\nu' - x Z_{\nu-1}' - Z_{\nu-1} = 0$$

Multiply by x: $x^2 Z_\nu'' + (\nu+1) x Z_\nu' - x^2 Z_{\nu-1}' - x Z_{\nu-1} = 0$

Scale (1) by ν: $\nu x Z_\nu' - \nu x Z_{\nu-1} + \nu^2 Z_\nu = 0$

subtract: $x^2 Z_\nu'' + x Z_\nu' + (\nu-1) x Z_{\nu-1} - \nu^2 Z_\nu - x^2 Z_{\nu-1}' = 0$

We want to eliminate $Z_{\nu-1}$ to obtain an equation only in Z_ν and use the recursion relation

$$J_{n+1} = \frac{n}{x} J_n - J_n' \text{ with } J_n = Z_n \text{ and } n = \nu - 1$$

$$\Rightarrow Z_\nu = \frac{(\nu-1)}{x} Z_{\nu-1} - Z_{\nu-1}'$$

so that $x Z'_{\nu-1} = (\nu-1) Z_{\nu-1} - x Z_\nu$

multiply by $x \Rightarrow x^2 Z_\nu = (\nu-1) x Z_{\nu-1} - x^2 Z'_{\nu-1}$

Substitute to obtain the resulting equation

$$\boxed{x^2 Z''_\nu + x Z'_\nu + (x^2 - \nu^2) Z_\nu = 0} \quad \text{Bessel D.E.}$$

It is common practice to scale the coordinate $x \to k\rho$

to obtain $\rho^2 \frac{d^2}{d\rho^2} Z_\nu(k\rho) + \rho \frac{d}{d\rho} Z_\nu(k\rho) + (k^2\rho^2 - \nu^2) Z_\nu(k\rho) = 0$

Integral representations

From the generating function $e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{m=-\infty}^{\infty} J_m(x) t^m$

we obtain $J_n(x)$ from the residue theorem as

$$\oint_C \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{n+1}} dt = \oint_C \sum_m J_m(x) t^{m-n-1} dt = 2\pi i J_n(x)$$

C encircles the singularity $t=0$

Take C as the unit circle and parameterize through $t = e^{i\vartheta}$

$dt = i e^{i\vartheta} d\vartheta$ and $e^{\frac{x}{2}(t-\frac{1}{t})} = e^{ix\sin\vartheta}$

$$\Rightarrow 2\pi i J_n(x) = \int_0^{2\pi} \frac{e^{ix\sin\vartheta}}{e^{(n+1)i\vartheta}} i e^{i\vartheta} d\vartheta = i \int_0^{2\pi} e^{i(x\sin\vartheta - n\vartheta)} d\vartheta$$

Assume x real and take the imaginary part of both sides

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(x\sin\vartheta - n\vartheta) d\vartheta = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\vartheta - n\vartheta) d\vartheta$$

n assumed integer

The real part gives $\int_0^{2\pi} \sin(x\sin\vartheta - n\vartheta) d\vartheta = 0$

Special case $n=0$ $\quad J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\vartheta) d\vartheta =$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{ix\cos\vartheta} d\vartheta$$

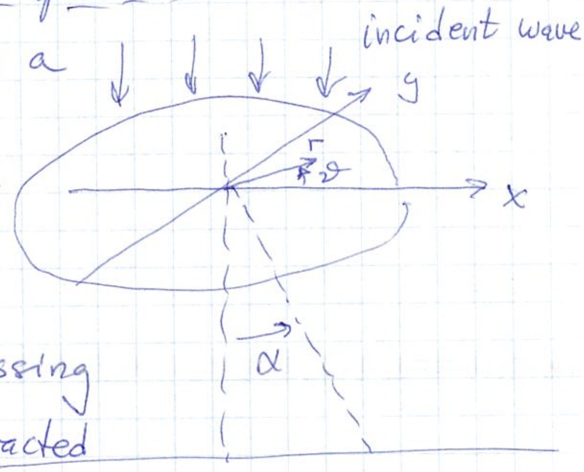
$$= \frac{1}{2\pi} \int_0^{2\pi} e^{ix\sin\vartheta} d\vartheta \quad =$$

Zeros of Bessel functions

The Bessel functions are oscillatory and their zeros give the nodes and scaling $x \to k\rho$ can be used to satisfy boundary conditions requiring a solution to vanish at the boundary, e.g. wave function of particle in sphere with completely repulsive walls or parallel component of the electric field at the walls of a cylindrical cavity with metallic walls.

Ex. Fraunhofer diffraction, circular aperture

wavelength λ, aperture of radius a



The amplitude of the diffracted wave is given by $\Phi \sim \int_0^a r dr \int_0^{2\pi} e^{ibr\cos\vartheta} d\vartheta$

$br\cos\vartheta$ is the phase of radiation passing (r, ϑ) in the aperture and ~~goin~~ diffracted angle α relative the incoming direction of propagation

Here $b = \frac{2\pi}{\lambda} \sin\alpha$

The integral is directly reduced to $\Phi \sim 2\pi \int_0^a J_0(br) r dr$

but we have $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ so

$$\Phi \sim 2\pi \int_0^a \frac{1}{b^2} \frac{d}{dr}\left[(br) J_1(br)\right] dr = \frac{2\pi}{b^2}\left[br J_1(br)\right]_0^a = \frac{2\pi a}{b} J_1(ab)$$

since $J_1(0) = 0$

The intensity is given by $\Phi^2 \sim \left(\frac{J_1\left[(2\pi a/\lambda)\sin\alpha\right]}{\sin\alpha} \right)^2$

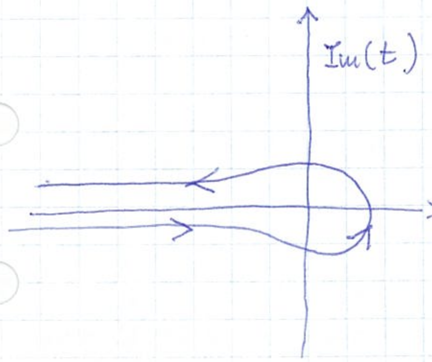
Minima at the zeros of J_1

i.e. $\frac{2\pi a}{\lambda} \sin\alpha = 3,8317, 7,0156, 10,1735, \ldots$

Nonintegral order Bessel functions
Schlaefli integral

Deform the circular contour valid for Bessel functions of integer order to stretch to infinity and open the contour there to introduce a branch cut to account for the branch point at $t=0$ for $t^{\nu+1}$ in

$$\boxed{F_\nu(x) = \frac{1}{2\pi i}\int_C \frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^{\nu+1}}\,dt}$$



$F_\nu(x)$ satisfies the Bessel equation

$$F_\nu' = \frac{1}{2\pi i}\int_C \frac{\frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^{\nu+1}}\,dt$$

$$F_\nu'' = \frac{1}{2\pi i}\int_C \frac{\frac{1}{4}\left(t-\frac{1}{t}\right)^2 e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^{\nu+1}}\,dt$$

$$x^2F_\nu'' + xF_\nu' + (x^2-\nu^2)F_\nu = \frac{1}{2\pi i}\int_C \frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^{\nu+1}}\left\{\frac{x^2}{4}\left(t-\frac{1}{t}\right)^2 + \frac{x}{2}\left(t-\frac{1}{t}\right) + x^2-\nu^2\right\}dt$$

but $$\frac{1}{2\pi i}\int_C \frac{d}{dt}\left\{\frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^\nu}\left[\nu+\frac{x}{2}\left(t+\frac{1}{t}\right)\right]\right\}dt =$$

$$= \frac{1}{2\pi i}\int_C e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\underbrace{\left\{\frac{\frac{x}{2}(1+t^{-2})\left[\nu+\frac{x}{2}\left(t+\frac{1}{t}\right)\right]}{t^\nu} + \frac{\frac{x}{2}\left(1-\frac{1}{t^2}\right)}{t^\nu} - \frac{\nu\left(\nu+\frac{x}{2}\left(t+\frac{1}{t}\right)\right)}{t^{\nu+1}}\right\}}_{\frac{\frac{x^2}{4}\left(t+\frac{1}{t}\right)^2+\frac{x}{2}\left(t-\frac{1}{t}\right)-\nu^2}{t^{\nu+1}} = \frac{\frac{x^2}{4}\left(t-\frac{1}{t}\right)^2+\frac{x}{2}\left(t-\frac{1}{t}\right)+x^2-\nu^2}{t^{\nu+1}}}dt$$

So the integrand is a derivative and, since the integration is within a region of analyticity we have

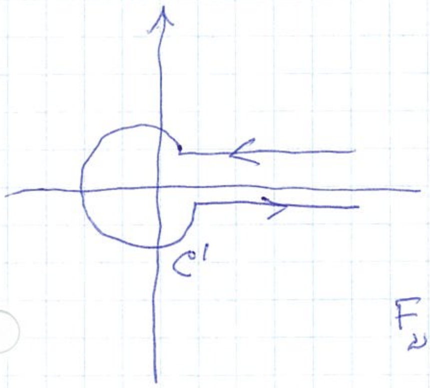
$$x^2F_\nu'' + xF_\nu' + (x^2-\nu^2)F_\nu =$$

$$= \frac{1}{2\pi i}\left\{\frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^\nu}\left[\nu+\frac{x}{2}\left(t+\frac{1}{t}\right)\right]\right\}_{\text{end}} - \frac{1}{2\pi i}\left\{\frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^\nu}\left[\nu+\frac{x}{2}\left(t+\frac{1}{t}\right)\right]\right\}_{\text{start}}$$

Both start and end of the open contour are at $-\infty$ so that for x positive this becomes zero

To show that F_ν is the solution called J_ν consider its value for small $x > 0$. Deform the contour to a large open circle and transform using $u = e^{-i\pi} \cdot \frac{xt}{2}$ to make the contour that of the gamma function

Note sign! $e^{-i\pi}\frac{xt}{2}$



$$u = e^{-i\pi}\frac{xt}{2} \to t = -\frac{2u}{x} \quad \text{and}$$

$$\frac{x}{2}\left(t - \frac{1}{t}\right) = \frac{x}{2}\left(-\frac{2u}{x} + \frac{x}{2u}\right) = -u + \frac{x^2}{4u} \approx -u$$

$$t^{\nu+1} = e^{-i\pi(\nu+1)}\left(\frac{2}{x}\right)^{\nu+1} u^{\nu+1}, \quad dt = \frac{2e^{-i\pi}}{x}du$$

$$F_\nu(x) \approx \frac{1}{2\pi i}\int_{C'} \frac{e^{-u}\frac{2e^{-i\pi}}{x}du}{e^{-i\pi(\nu+1)}\left(\frac{2}{x}\right)^{\nu+1}u^{\nu+1}} =$$

$$= \frac{1}{2\pi i}\left(\frac{x}{2}\right)^{\nu} e^{+i\nu\pi} \underbrace{\int_{C'} e^{-u}u^{-\nu-1}du}_{2ie^{-i(\nu+1)\pi}\Gamma(-\nu)\sin[-(\nu+1)\pi]} =$$

$$= 2ie^{-(\nu+1)i\pi}\Gamma(-\nu)\sin\nu\pi$$

So that $F_\nu(x) \approx \left(\frac{x}{2}\right)^{\nu} e^{-i\pi}\frac{\Gamma(-\nu)\sin\nu\pi}{\pi} =$

$$= \left(\frac{x}{2}\right)^{\nu}\frac{\Gamma(-\nu)\sin(-\nu\pi)}{\pi} = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}$$

which is the small-x behavior of $J_\nu(x)$

Orthogonality -

Restructure the Bessel equation as a Sturm - Liouville problem

$$\underbrace{-\left(\frac{d^2}{d\varrho^2}+\frac{1}{\varrho}\frac{d}{d\varrho}-\frac{\nu^2}{\varrho^2}\right)}_{\mathcal{L}} Z_\nu(k\varrho) = k^2 Z_\nu(k\varrho)$$

Consider $Z_\nu(k\varrho) = J_\nu(k\varrho)$; The weight factor that makes $\mathcal{L}$ self-adjoint is ϱ $\left(P_0(x) = -1,\ P_1(x) = \frac{-1}{\varrho},\ w(x) = \frac{1}{P_0}\exp\left\{\int^x \frac{P_1}{P_0} ds\right\}\right.$

The orthogonality integral $\int_0^a \varrho J_\nu(k\varrho) J_\nu(k'\varrho)\, d\varrho =$

$$= \frac{1}{k^2-k'^2}\Big[\varrho\left(k' J_\nu(k\varrho) J_\nu'(k'\varrho) - k J_\nu'(k\varrho) J_\nu(k'\varrho)\right)\Big]_0^a =$$

(*) $$= \frac{1}{k^2-k'^2}\Big[a\left(k' J_\nu(ka) J_\nu'(k'a) - k J_\nu'(ka) J_\nu(k'a)\right)\Big] = 0 \quad \text{if } k \neq k'$$

and k, k' chosen so that $J_\nu(ka) = J_\nu(k'a) = 0$

Normalization: Take the limit $k \to k'$ ~~of~~ of (*) using L'Hôpital's rule

$$\int_0^a \varrho\left[J_\nu(k\varrho)\right]^2 d\varrho = \lim_{k'\to k} \frac{a\left\{J_\nu(ka)\frac{d}{dk'}\left(k' J_\nu'(k'a)\right) - k J_\nu'(ka)\frac{d}{dk'}\left(J_\nu(k'a)\right)\right\}}{\frac{d}{dk'}\left(k^2-k'^2\right)}$$

Let $ka = \alpha_{\nu i}$ be a zero of J_ν so that $k = \frac{\alpha_{\nu i}}{a}$

Then $$\int_0^a \varrho\left[J_\nu\left(\alpha_{\nu i}\frac{\varrho}{a}\right)\right]^2 d\varrho = \frac{-ka^2\left[J_\nu'(ka)\right]^2}{-2k} = \frac{a^2}{2}\left[J_\nu'(\alpha_{\nu i})\right]^2$$

but $J_\nu(x) = -J_{\nu+1}'(x) + \frac{\nu-1}{x} J_{\nu-1}$

set $\nu \to \nu+1$, $x \to \alpha_{\nu i}$ $\Rightarrow$ $J_\nu'(\alpha_{\nu i}) = -J_{\nu+1}(\alpha_{\nu i})$

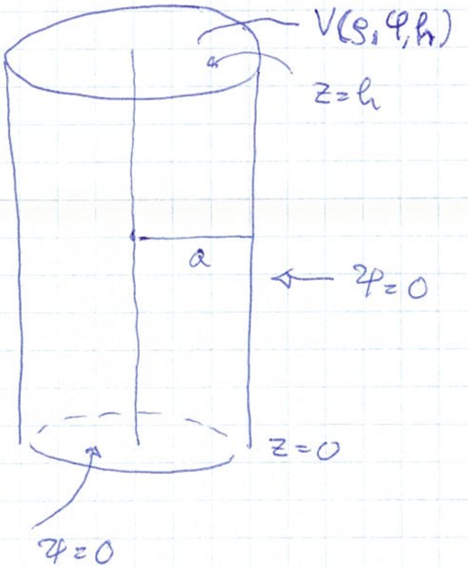
$$\therefore \int_0^a \varrho\left[J_\nu\left(\alpha_{\nu i}\frac{\varrho}{a}\right)\right]^2 d\varrho = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu i})\right]^2$$

With the orthogonality and normalization of the Bessel functions $J_\nu(\alpha_{\nu j}\,\varrho/a)$ for fixed ν and $j = 1, 2, 3, \ldots$ and the assumption of completeness then

$$f(\varrho) = \sum_{j=1}^{\infty} c_{\nu j}\, J_\nu\left(\alpha_{\nu j}\frac{\varrho}{a}\right), \quad 0 \le \varrho \le a \;, \;\nu > -1$$

for any well-behaved function $f(\varrho)$ defined on the same interval. The coefficients $c_{\nu j}$ are found through projection

$$c_{\nu j} = \frac{2}{a^2\left[J_{\nu+1}(\alpha_{\nu j})\right]^2}\int_0^a f(\varrho)\, J_\nu\left(\alpha_{\nu j}\frac{\varrho}{a}\right)\varrho\, d\varrho$$



Electrostatic potential in hollow cylinder

Separate variables in Laplace equation in cylindrical coordinates to obtain

$$\psi_{\ell m}(\varrho, \varphi, z) = P_{\ell m}(\varrho)\, \Phi_m(\varphi)\, Z_\ell(z)$$

where $\Phi_m(\varphi) = e^{\pm i m \varphi}$

$$\frac{d^2 Z_\ell}{dz^2} = \ell^2 Z_\ell(z) \;; \quad Z_\ell(z) = e^{\pm \ell z}$$

$$\varrho^2 \frac{d^2 P_{\ell m}}{d\varrho^2} + \varrho \frac{d}{d\varrho} P_{\ell m} + (\ell^2 \varrho^2 - m^2) P_{\ell m} = 0$$

$P_{\ell m}$ are Bessel functions $J_m(\ell\varrho)$ with $\ell = \alpha_{mj}/a$ to satisfy boundary condition $\psi(a, \varphi, z) = 0 \quad (0 \le z < h)$

For Z_ℓ choose $Z_\ell(z) = \sinh(\ell z)$ to satisfy BC at $z = 0$

Solutions $\psi_{mj} = c_{mj}\, J_m\left(\alpha_{mj}\frac{\varrho}{a}\right) e^{im\varphi} \sinh\left(\alpha_{mj}\frac{z}{a}\right)$

At $z = h$

$$V(\varrho, \varphi, z) = \sum_{m=-\infty}^{\infty}\sum_{j=1}^{\infty} \psi_{mj}\; c_{mj}$$

At $z = h$ $\quad V(\varrho, \varphi, h) = \sum_{m=-\infty}^{\infty}\sum_{j=1}^{\infty} c_{mj}\, J_m\left(\alpha_{mj}\frac{\varrho}{a}\right) e^{im\varphi} \sinh\left(\alpha_{mj}\frac{h}{a}\right)$

Find coefficients through projection both on the trigonometric series and the Bessel functions using

$$\int_0^{2\pi} e^{-im\varphi} e^{im'\varphi} d\varphi = \delta_{mm'} \cdot 2\pi$$

$$c_{mj} = \left\{ \pi a^2 \sinh\left(\alpha_{mj} \frac{h}{a}\right) J_{m+1}^2(\alpha_{mj}) \right\}^{-1} \cdot \int_0^{2\pi} d\varphi \int_0^a V(\rho,\varphi,h) J_m\left(\alpha_{mj} \frac{\rho}{a}\right) e^{-im\varphi} \rho \, d\rho$$

Bessel functions of the second kind (Neumann functions)

$$Y_\nu(x) = \frac{\cos\nu\pi \, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \qquad (6^{th} \text{ edition } N_\nu \equiv Y_\nu)$$

For nonintegral ν this solves the Bessel equation since J_ν, $J_{-\nu}$ do. The behavior for small x (noninteger ν):

$$Y_\nu(x) = \frac{\cos\nu\pi \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(\nu+s+1)} \left(\frac{x}{2}\right)^{\nu+2s} - \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(-\nu+s+1)} \left(\frac{x}{2}\right)^{-\nu+2s}}{\sin\nu\pi} =$$

$$= -\frac{1}{\sin\nu\pi} \left\{ \frac{1}{\Gamma(1-\nu)} \left(\frac{x}{2}\right)^{-\nu} - \dots \right\}$$

The reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$

$$\Rightarrow \sin\nu\pi \, \Gamma(1-\nu) = \frac{\pi}{\Gamma(\nu)}$$

$$Y_\nu(x) = -\frac{\Gamma(\nu)}{\pi} \left(\frac{x}{2}\right)^{-\nu} + \dots \qquad \text{irregular solution at } x=0$$

For ν integer L'Hôpital's rule is applied

$$Y_n(x) = \left. \frac{\frac{d}{d\nu}\{\cos\nu\pi \, J_\nu - J_{-\nu}\}}{\frac{d}{d\nu}\sin\nu\pi} \right|_{\nu=n} = \frac{1}{\pi} \left\{ \frac{dJ_\nu}{d\nu} - (-1)^n \frac{dJ_{-\nu}}{d\nu} \right\}_{\nu=n}$$

$$\Rightarrow Y_n(x) = \frac{2}{\pi} J_n(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} -$$

$$- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left\{ \psi(k+1) + \psi(n+k+1) \right\} \left(\frac{x}{2}\right)^{2k+n}$$

$$\psi(k+1) = -\gamma + H_k = -\gamma + \sum_{m=1}^{k} \frac{1}{m} \qquad \text{digamma function}$$

$$\psi(z+1) = [\Gamma(z+1)]' / \Gamma(z+1)$$

$Y_n(x)$ contain the term $\ln\left(\frac{x}{2}\right)$ times the regular solution
$Y_n(x)$ irregular at the origin and are oscillatory for Larger x.

Recurrence relations:

we have $J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu$

and $J_{-\nu-1} + J_{-\nu+1} = -\frac{2\nu}{x} J_{-\nu}$

So that $Y_{\nu-1} + Y_{\nu+1} = \frac{\cos[(\nu-1)\pi] J_{\nu-1} - J_{-\nu+1}}{\sin(\nu-1)\pi} +$

$$+ \frac{\cos[(\nu+1)\pi] J_{\nu+1} - J_{-\nu-1}}{\sin[(\nu+1)\pi]} =$$

$$\cos[(\nu+1)\pi] = \cos\nu\pi \cos\pi = -\cos\nu\pi$$

$$\sin[(\nu+1)\pi] = \sin\nu\pi \cos\pi = -\sin\nu\pi$$

$$\cos[(\nu-1)\pi] = \cos\nu\pi \cos\pi = -\cos\nu\pi$$

$$\sin[(\nu-1)\pi] = \sin\nu\pi \cos\pi = -\sin\nu\pi$$

$$= \frac{\cos\nu\pi\,(J_{\nu-1} + J_{\nu+1})}{\sin\nu\pi} + \frac{J_{-\nu+1} + J_{-\nu-1}}{\sin\nu\pi} =$$

$$= \frac{\cos[\nu\pi]\,\frac{2\nu}{x} J_\nu}{\sin\nu\pi} + \frac{\left(-\frac{2\nu}{x}\right) J_{-\nu}}{\sin\nu\pi} = \frac{2\nu}{x} Y_\nu$$

Y_ν satisfy the same recurrence relations as J_ν and thus also the Bessel D.E.

Wronskian formulas

An ODE $p(x)\,y'' + q(x)\,y' + r(x)\,y = 0$ on self-adjoint form ($q(x) = \frac{d}{dx}p(x)$) has the Wronskian connecting two linearly independent solutions u and v

$$u(x)\,v'(x) - u'(x)\,v(x) = \frac{A}{p(x)}$$

Divide the Bessel equation $x^2 y'' + x y' + (x^2 - \nu^2)\,y = 0$
by x to obtain $x y'' + y' + (x - \frac{\nu^2}{x})\,y = 0$
so that $p(x) = x$

then for non integer ν $J_\nu J_{-\nu}' - J_\nu' J_{-\nu} = \frac{A_\nu}{x}$

since A_ν is a constant it can be identified by considering the Wronskian for $x \to 0$

We have $J_\nu \to \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^{\nu}$ $\quad J_\nu' \to \frac{\nu}{2\Gamma(1+\nu)}\left(\frac{x}{2}\right)^{\nu-1}$

$J_{-\nu} \to \frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu}$ $\quad J_{-\nu}' \to \frac{-\nu}{2\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu-1}$

So that

$$J_\nu J_{-\nu}' - J_\nu' J_{-\nu} = \frac{-\nu}{2\Gamma(1+\nu)\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-1} - \frac{\nu}{2\Gamma(1+\nu)\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-1}$$

$$= \frac{-2\nu}{x\,\Gamma(1+\nu)\Gamma(1-\nu)} = -\frac{2\nu}{x\,\nu\,\Gamma(\nu)\Gamma(1-\nu)} = -\frac{2\sin\nu\pi}{\pi x}$$

using $\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin\nu\pi}$

$$\therefore A_\nu = -\frac{2\sin[\nu\pi]}{\pi}$$

Alternative forms from recurrence relations

$$J_\nu J_{-\nu+1} + J_{-\nu} J_{\nu-1} = \frac{2\sin\nu\pi}{\pi x} \qquad (1)$$

$$J_\nu J_{-\nu-1} + J_{-\nu} J_{\nu+1} = -\frac{2\sin\nu\pi}{\pi x} \qquad (2)$$

$$J_\nu Y_\nu' - J_\nu' Y_\nu = \frac{2}{\pi x} \qquad (3)$$

$$J_\nu Y_{\nu+1} - J_{\nu+1} Y_\nu = -\frac{2}{\pi x} \qquad (4)$$

Example (1) First show $J_\nu J_{-\nu+1} + J_{-\nu} J_{\nu-1}$ is a Wronskian

From $J_n'(x) = \pm J_{n\pm1} + \frac{n\pm1}{x} J_{n\pm1}$

$$J_{-\nu+1} = -J_{-\nu}' + \frac{-\nu}{x} J_{-\nu}$$

$$J_{\nu-1} = J_\nu' + \frac{\nu}{x} J_\nu$$

so $J_\nu J_{-\nu+1} + J_{-\nu} J_{\nu-1} = J_\nu\left(-J_{-\nu}' - \frac{\nu}{x} J_{-\nu}\right) + J_{-\nu}\left(J_\nu' + \frac{\nu}{x} J_\nu\right) =$

$$= -J_\nu J_{-\nu}' - \frac{\nu}{x} J_\nu J_{-\nu} + J_{-\nu} J_\nu' + \frac{\nu}{x} J_\nu J_{-\nu} =$$

$$= J_\nu' J_{-\nu} - J_\nu J_{-\nu}' \quad \text{a Wronskian}$$

For small x $J_\nu(x) \sim x^\nu$ but $J_{-\nu} \sim x^{-\nu}$ so the first term ($J_\nu J_{-\nu+1}$) is neglected; the second term dominates and $J_{-\nu} J_{\nu-1} = \frac{x^{-\nu}}{2^{-\nu}\Gamma(-\nu+1)} \cdot \frac{x^{\nu-1}}{2^{\nu-1}\Gamma(\nu)} = \frac{2}{x\Gamma(\nu)\Gamma(1-\nu)}$

$$= \frac{2\sin[\nu\pi]}{\pi x}$$

For a coaxial wave guide $\rho = 0$ is excluded and we will get both Bessel and Neumann solutions ($Y_\nu(k\rho)$)