Chapter 13 Gamma function

We'll use three definitions to determine properties

1) Infinite limit (Euler)

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)}\, n^{z}, \quad z \neq 0, -1, -2, \ldots$$

Consider $\Gamma(z+1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)(z+3)\cdots(z+n+1)}\, n^{z+1} =$

$$= \lim_{n\to\infty} \frac{nz}{z+n+1} \cdot \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)\cdots(z+n)}\, n^{z} = z\Gamma(z)$$

Then $\Gamma(1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n(n+1)}\, n = 1$

Then $\Gamma(2) = 1\cdot\Gamma(1) = 1$

$\Gamma(3) = 2\cdot\Gamma(2) = 1\cdot 2$, $\Gamma(4) = 3\Gamma(3) = 1\cdot 2\cdot 3$

and in general $\Gamma(n) = 1\cdot 2\cdot 3\cdots(n-1) = (n-1)!$

2) Definite integral (Euler)

$$\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \mathrm{Re}(z) > 0$$

transform using $t = u^2$, $dt = 2u\,du$

$$\Gamma(z) = \int_0^\infty e^{-u^2} u^{2(z-1)} \cdot 2u\,du = 2\int_0^\infty e^{-u^2} u^{2z-1}\, du$$

transform using $t = -\ln u$, $dt = -\frac{du}{u}$, $t=0 \to u=1$, $t=\infty \to u=0$

$$\Gamma(z) = \int_1^0 e^{\ln u}\left[\ln\left(\frac{1}{u}\right)\right]^{z-1}\left(-\frac{du}{u}\right) = \int_0^1 \left[\ln\left(\frac{1}{u}\right)\right]^{z-1} du$$

Consider $\Gamma(z) = 2\int_0^\infty e^{-u^2} u^{2z-1}\, du$ for argument $z = \frac{1}{2}$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\, du = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

The definite integral and infinite limit definitions are equivalent

Write e^{-t} as the limit $\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n = e^{-t}$ and consider the function $F(z,n) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt$, $Re(z)>0$

Taking the limit $n\to\infty$ gives $\Gamma(z)$ through

$$\lim_{n\to\infty} F(z,n) = F(z,\infty) = \int_0^\infty e^{-t}t^{z-1}dt \equiv \Gamma(z)$$

Substitute $u=\frac{t}{n}$ in $F(z,n)$ with $t=nu$, $dt=ndu$, $t=0\to u=0$, $t=n\to u=1$

$$F(z,n) = \int_0^1 (1-u)^n (nu)^{z-1}\cdot ndu = n^z\int_0^1 (1-u)^n u^{z-1}du$$

Integrate by parts: $\frac{F(z,n)}{n^z} = \underbrace{\left[(1-u)^n\frac{u^z}{z}\right]_0^1}_{\substack{=0\\ Re(z)>0}} + \frac{n}{z}\int_0^1 (1-u)^{n-1}u^z du =$

$$= \dots = \frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)}\int_0^1 u^{z+n-1}du$$

$\Rightarrow F(z,n) = \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)\cdots(z+n-1)(z+n)}n^z$ and $\lim_{n\to\infty}F(z,n) = F(z,\infty)\equiv\Gamma(z)$

3) Infinite product (Weierstrass)

$$\frac{1}{\Gamma(z)} \equiv z e^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

with γ the Euler-Mascheroni constant $\gamma = 0.5772156619\dots$

Derivation from the infinite limit:

Write $\Gamma(z) = \lim_{n\to\infty}\frac{1\cdot 2\cdot 3\cdots n}{z(z+1)\cdots(z+n)}n^z = \lim_{n\to\infty}\frac{1}{z}\prod_{m=1}^{n}\left(1+\frac{z}{m}\right)^{-1}n^z$

Take $\frac{1}{\Gamma(z)}$ and use $n^{-z} = e^{(-\ln n)z}$ to obtain

$$\frac{1}{\Gamma(z)} = z\lim_{n\to\infty}e^{(-\ln n)z}\prod_{m=1}^{n}\left(1+\frac{z}{m}\right)$$

We have $\exp\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)z\right\} = \prod_{m=1}^{n}e^{z/m}$ $(*)$

The Euler-Mascheroni constant is the limit

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \ln n\right) \qquad \text{(Eq. 1.13)}$$

Multiply with the left hand side of (*) and divide by the right hand side to get:

$$\frac{1}{\Gamma(z)} = z\left\{\lim_{n\to\infty} \exp\left[\left(\underbrace{1+\tfrac{1}{2}+\tfrac{1}{3}+\dots+\tfrac{1}{n}-\ln n}_{\gamma}\right)z\right]\right\}\cdot\left[\lim_{n\to\infty}\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)e^{-z/m}\right]$$

x ———————————————————————————— x

Euler's reflection formula: $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

The approach in the book leading to the integral $\int_0^\infty \frac{v^z dv}{(v+1)^2}$ requires that $|z|<1$. Instead we have the product form for $\sin z$ as $\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$ so $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$

Take $$\frac{1}{-z\Gamma(z)\Gamma(-z)} = -\frac{1}{z}\cdot z e^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\cdot(-z)e^{-\gamma z}\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}$$

$$= z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right) = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) =$$

$$= \frac{\sin \pi z}{\pi}$$

$$\therefore\ -z\Gamma(z)\Gamma(-z) = \frac{\pi}{\sin \pi z}$$

and from the difference equation $\Gamma(z+1) = z\Gamma(z)$ with $-z$ $\Gamma(1-z) = -z\,\Gamma(-z)$

$$\therefore\ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

For $z=\frac{1}{2}$ we get $\left[\Gamma(\tfrac{1}{2})\right]^2 = \pi$ and $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

Legendre's duplication formula:

$$\Gamma(1+z)\Gamma(z+\tfrac{1}{2}) = 2^{-2z}\sqrt{\pi}\ \Gamma(2z+1)$$

Special case $z=n$, positive integer

$\Gamma(1+n) = n!$, $\Gamma(2n+1) = (2n)!$

$$\Gamma(n+\tfrac{1}{2}) = \Gamma(\tfrac{1}{2})\cdot\left[\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\right] = \sqrt{\pi}\,\frac{1\cdot 3\cdot 5\cdots 2n-1}{2^n} = \sqrt{\pi}\,\frac{(2n-1)!!}{2^n}$$

Now $n! = \dfrac{(2n)!!}{2^n}$ and

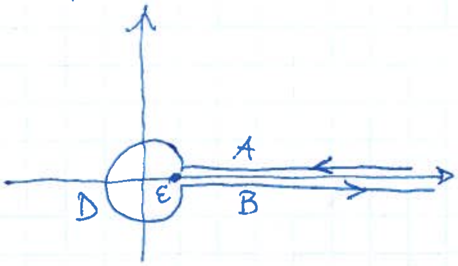
$(2n)! = (2n)!! \cdot (2n-1)!!$

So that $\Gamma(1+n)\Gamma(n+\frac{1}{2}) = n!\sqrt{\pi}\,\frac{(2n-1)!!}{2^n} = \sqrt{\pi}\,\frac{(2n)!!}{2^n}\,\frac{(2n-1)!!}{2^n} =$

$= \sqrt{\pi}\, 2^{-2n}(2n)! = \sqrt{\pi}\, 2^{-2n}\,\Gamma(2n+1)$

Schlaefli integral:

open contour



$\int_C e^{-t} t^{\nu} dt = (e^{2\pi i\nu} - 1)\Gamma(\nu+1)$ ν non-integer

$t=0$ is a branchpoint

cannot close at $z=+\infty$ because of the cut

cannot close with large circle since e^{-t} becomes infinite ~~for~~ in the limit of large negative t.

The integral along A from $+\infty$ to ε gives $-\Gamma(\nu+1)$ taking $\arg(z) = 0$

The integral along B from ε to $+\infty$ is in the fourth quadrant gives $e^{2\pi i\nu}\Gamma(\nu+1)$

For $\nu > -1$ the circle gives zero contribution so

(for $\nu+1>0$) $\int_C e^{-t} t^{\nu} dt = (e^{2\pi i\nu} - 1)\Gamma(\nu+1) = 2i\, e^{i\nu\pi}\,\Gamma(\nu+1)\sin\nu\pi$

x -- x

Maxwell-Boltzmann distribution

A structure less ideal gas has density of states $n(E)$ proportional to $E^{1/2}$ where E is the kinetic energy of a molecule $0 \le E < \infty$

A state is occupied with probability $ce^{-E/kt} \equiv ce^{-\beta E}$

The number of states in $[E, E+dE]$ is $n(E)dE$

Total probability of occupancy in any state is

$$1 = C\int_0^{\infty} n(E)\, e^{-\beta E} dE$$

Average energy is $\langle E \rangle = C\int_0^{\infty} E\, n(E)\, e^{-\beta E} dE$

Ideal structureless gas

Find the normalization constant C

$$z-1=\tfrac{1}{2} \rightarrow z=\tfrac{3}{2}$$

$$1 = C\int_0^{\infty} E^{1/2} e^{-\beta E}\, dE = \begin{bmatrix} \beta E = t \\ E = t/\beta \\ dE = \frac{dt}{\beta} \end{bmatrix} = \frac{C}{\beta^{3/2}} \int_0^{\infty} t^{1/2} e^{-t}\, dt =$$

$$= \frac{C}{\beta^{3/2}}\Gamma\left(\frac{3}{2}\right) = \frac{C\sqrt{\pi}}{2\beta^{3/2}} \rightarrow C = \frac{2\beta^{3/2}}{\sqrt{\pi}}$$

$$z-1=\tfrac{3}{2} \rightarrow z=\tfrac{5}{2}$$

Mean energy $\langle E\rangle = C\int_0^{\infty} E^{3/2} e^{-\beta E}\, dE = \frac{C}{\beta^{5/2}}\int_0^{\infty} t^{3/2} e^{-t}\, dt =$

$$= \frac{2\beta^{3/2}}{\sqrt{\pi}} \cdot \frac{1}{\beta^{5/2}} \cdot \Gamma\left(\frac{5}{2}\right) = \frac{2}{\sqrt{\pi}\,\beta} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} = \frac{3}{2}kT$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\left\{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right\} = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

(Digamma function $\frac{d}{dz}\ln\Gamma(z+1) \equiv \psi(z+1) = \lim_{n\to\infty}\left(\ln n - \frac{1}{z+1} - \frac{1}{z+2} - \dots - \frac{1}{z+n}\right)$

$$= \frac{\{\Gamma(z+1)\}'}{\Gamma(z+1)}$$

and $\psi(n+1) = -\gamma + \sum_{m=1}^{n} \frac{1}{m}$)