Infinite products

We have the infinite product forms of $\sin z$ and $\cos z$ as

$$\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right) \quad \text{and} \quad \cos z = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{(n-\frac{1}{2})^2\pi^2}\right)$$

Infinite products are related to infinite sums by taking the logarithm:

$$P=\prod_{n=1}^{\infty}(1+a_n) \rightarrow \ln\left(\prod_{n=1}^{\infty}(1+a_n)\right)=\sum_{n=1}^{\infty}\ln(1+a_n)$$

Convergence: If $0\le a_n<1$, the infinite products $\prod_{n=1}^{\infty}(1+a_n)$ and $\prod_{n=1}^{\infty}(1-a_n)$ converge if $\sum_{n=1}^{\infty}a_n$ converges and diverge if the sum $\sum_{n=1}^{\infty}a_n$ diverges.

Proof: In $\prod_{n=1}^{\infty}(1+a_n)$ we have for the individual terms that

$$e^{a_n}=1+a_n+\frac{a_n^2}{2!}+\frac{a_n^3}{3!}+\dots \ge 1+a_n \qquad (a_n\ge 0)$$

so that $\prod_{n=1}^{\infty}(1+a_n)\le \exp\left\{\sum_{n=1}^{\infty}a_n\right\}$ gives upper bound

A lower bound is found from the partial product

$$P_n=\prod_{i=1}^{n}(1+a_i)=1+\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}\sum_{j=1}^{n}a_ia_j+\dots \ge \sum_{i=1}^{n}a_i$$

Let $n\to\infty$ $\quad \prod_{i=1}^{\infty}(1+a_i)\ge\sum_{i=1}^{\infty}a_i$

For $\prod_{n=1}^{\infty}(1-a_n)$ the proof in terms of the infinite sum is more complicated. Taken at face value with $0\le a_n<1$ the convergence is trivial since each term in the product is bounded by 1. However it becomes important when considering the sine and cosine infinite product expansions setting for the $\sin(z)$ function $a_n=\frac{z^2}{n^2\pi^2}$ giving $\sum_{n=1}^{\infty}a_n=\frac{z^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$

In chapter 11 we evaluated the sum

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\frac{\pi\coth a\pi}{2a}-\frac{1}{2a^2}$$

; Take the limit $a\to 0$

The Taylor expansion of $\coth(x) = \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + , \ldots$
so that

$$\frac{\pi \coth \pi a}{2a} - \frac{1}{2a^2} = \frac{\pi}{2a}\left\{\frac{1}{\pi a} + \frac{\pi a}{3} - \frac{1}{45}(\pi a)^3 + \ldots\right\} - \frac{1}{2a^2} =$$

$$= \frac{\pi^2}{6} - O(a^2)$$

$$\lim_{a\to 0}\left\{\frac{\pi \coth \pi a}{2a} - \frac{1}{2a^2}\right\} = \frac{\pi^2}{6}$$ so that $\frac{z^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{z^2}{6}$ and the product expansion of $\sin z$ converges for all z.

Similarly for $\cos(z)$ $\quad a_n = \frac{z^2}{(n-\frac{1}{2})^2\pi^2} = \frac{4z^2}{\pi^2}\cdot\frac{1}{(2n-1)^2}$

and $\sum_{n=1}^{\infty} a_n = \frac{4z^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{4z^2}{\pi^2}\cdot\frac{\pi^2}{8} = \frac{z^2}{2}$ also convergent

x———x

Evaluate $P = \prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right)$: This corresponds to the value of

$$\frac{\sin z}{z\left(1 - \frac{z^2}{\pi^2}\right)} = \prod_{n=2}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$$ at $z = \pi$;

L'Hôpital's rule gives

$$P = \lim_{z\to\pi}\frac{\frac{d}{dz}\sin z}{\frac{d}{dz}\left(z - \frac{z^3}{\pi^2}\right)} = \lim_{z\to\pi}\frac{\cos z}{1 - \frac{3z^2}{\pi^2}} = \frac{1}{2}$$

Asymptotic series

Consider the exponential integral function

$$Ei(x) = \int_{-\infty}^{x}\frac{e^u}{u}du$$

(Appears in astrophysics when dealing with a gas with Maxwell-Boltzmann energy distribution)

rewrite as $\quad -Ei(-x) = \int_x^{\infty}\frac{e^{-u}}{u}du \equiv E_1(x)$

$$\left(E_i(-x) = \int_{-\infty}^{-x}\frac{e^u}{u}du = -\int_{-x}^{-\infty}\frac{e^u}{u}du = \left[u = -v\right] = -\int_x^{\infty}\frac{e^{-v}}{v}dv \right)$$

We need to evaluate $E_1(x)$ for large values of x

Generalize to $I(x,p) = \int_x^{\infty}\frac{e^{-u}}{u^p}du$

Integrate by parts:

$$I(x,p) = \frac{e^{-x}}{x^p} - p\int_x^\infty \frac{e^{-u}}{u^{p+1}}du = \frac{e^{-x}}{x^p} - \frac{pe^{-x}}{x^{p+1}} + p(p+1)\int_x^\infty \frac{e^{-u}}{u^{p+2}}du$$

continue to obtain

$$I(x,p) = e^{-x}\left\{\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}} - \dots + (-1)^{n-1}\frac{(p+n-2)!}{(p-1)!\,x^{p+n-1}}\right) +$$

$$+ (-1)^n\frac{(p+n-1)!}{(p-1)!}\int_x^\infty \frac{e^{-u}}{u^{p+n}}du$$

The series diverges for all finite values of x:

d'Alembert ratio test: $\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty}\frac{(p+n)!}{(p+n-1)!}\cdot\frac{1}{x} = \lim_{n\to\infty}\frac{p+n}{x} \to \infty$

The partial sums still approximate the value. Investigate the remainder $R_n(x,p)$ as

$$I(x,p) - s_n(x,p) = (-1)^{n+1}\frac{(p+n)!}{(p-1)!}\int_x^\infty \frac{e^{-u}}{u^{p+n+1}}du \equiv R_n(x,p)$$

Move the dependency on x from the integral bound to the integrand: $u = v + x$ so that

$$\int_x^\infty \frac{e^{-u}}{u^{p+n+1}}du = e^{-x}\int_0^\infty \frac{e^{-v}}{(v+x)^{p+n+1}}dv = \frac{e^{-x}}{x^{p+n+1}}\underbrace{\int_0^\infty \frac{e^{-v}dv}{\left(1+\frac{v}{x}\right)^{p+n+1}}}_{\xrightarrow[x\to\infty]{} 1}$$

For large x the last integral approaches 1 and

$$|R_n(x,p)| \approx \frac{(p+n)!}{(p-1)!}\frac{e^{-x}}{x^{p+n+1}}$$ so that taking

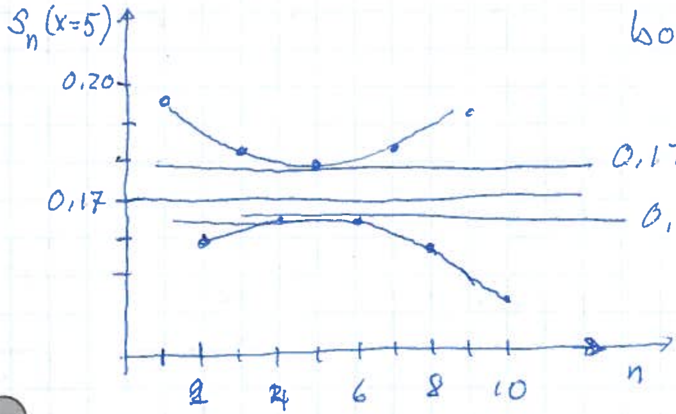
x large enough the partial sum $s_n(x,p)$ will be an arbitrarily good approximation to $I(x,p)$

We have $e^x E_1(x) = e^x\int_x^\infty \frac{e^{-u}}{u}du \approx S_n(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots$

$$\dots + (-1)^n\frac{n!}{x^{n+1}}$$

The series must be terminated after some n

Now the remainder alternates in sign meaning that successive partial sums alternate between lower and upper bounds. The sum is semi-convergent and the value bounded by the closest approach



$$0{,}1664 \leq e^x E_1(x)\Big|_{x=5} \leq 0{,}1741$$

Actual value $e^x E_1(x)\Big|_{x=5} = 0{,}1704$

Cosine and sine integrals

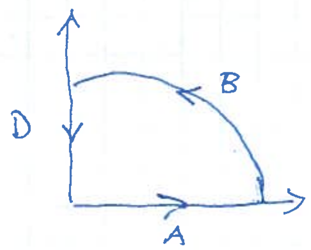
$$Ci(u) = -\int_u^\infty \frac{\cos t}{t}\,dt \qquad si(u) = -\int_u^\infty \frac{\sin t}{t}\,dt$$

Combine to give e^{it}: $Ci(u) + i\,si(u) = -\int_u^\infty \frac{e^{it}}{t}\,dt$

Transform $t = u + z$ to remove lower bound

$$F(u) = Ci(u) + i\,si(u) = -e^{iu}\int_0^\infty \frac{e^{iz}\,dz}{u+z}$$ to be evaluated for large positive real u

The integrand is oscillatory. Consider the contour integral $-e^{iu}\oint_C \frac{e^{iz}\,dz}{u+z}$ where C is the arc in the first quadrant



The segment A is the desired integral. The arc along B does not contribute due to the exponential and denominator. No poles are contained so the integral is the negative of the contribution from segment D

Along D $z = iy$ and $I_D = -e^{iu}\int_\infty^0 \frac{e^{-y}\,i\,dy}{u+iy} = e^{iu}\int_0^\infty \frac{e^{-y}\,i\,dy}{u+iy}$

Exponentially decaying integrand

So $F(u) = -e^{iu}\int_0^\infty \frac{e^{-y}\, i\, dy}{u+iy}$

Expand $\frac{1}{u+iy} = \frac{1}{u}\frac{1}{1+\frac{iy}{u}} = \frac{1}{u}\left\{1 - \frac{iy}{u} + \left(\frac{iy}{u}\right)^2 - \dots\right\}$ divergent for $y > u$

Insert and integrate termwise using $\int_0^\infty y^n e^{-y}\, dy = n!$

Which gives $F(u) \approx -\frac{ie^{iu}}{u}\left\{1 - i\left(\frac{1!}{u}\right) - \left(\frac{2!}{u^2}\right) + i\left(\frac{3!}{u^3}\right) + \frac{4!}{u^4} - \dots\right\}$

For u ~~[illegible]~~ sufficiently large the terms will initially decrease to small values before increasing towards divergence

To get Ci and si write $e^{iu} = \cos u + i\sin u$ and separate into real and imaginary parts

$$Ci(u) \approx \frac{\sin u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n)!}{u^{2n}} - \frac{\cos u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n+1)!}{u^{2n+1}}$$

$$si(u) \approx -\frac{\cos u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n)!}{u^{2n}} - \frac{\sin u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n+1)!}{u^{2n+1}}$$

General: Assume power series for simplicity

$$\underset{\text{remainder}}{x^n R_n(x)} = x^n\left[\underset{\text{function}}{f(x)} - \underset{\text{partial sum}}{s_n(x)}\right] \qquad s_n(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_n}{x^n}$$

The asymptotic expansion has the properties that

$$\lim_{x\to\infty} x^n R_n(x) = 0 \quad \text{for fixed } n$$

$$\lim_{n\to\infty} x^n R_n(x) = \infty \quad \text{for fixed } x$$

For a power series $R_n(x) \approx x^{-n-1}$

and $f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}$ with equality only in the limit $x \to \infty$ and a finite number of terms used

Method of Steepest Descents

Conditions: Asymptotic behavior at large, real t is needed for function $f(t)$

- $f(t) = \int_C F(z,t)\,dz$ with $F(z,t)$ analytic in z and parametrically dependent on t
- The path C is such that for large t the dominant contribution is from a small range z near z_0 where $|F(z_0,t)|$ is a maximum on the path.
- The path goes through z_0 in the orientation that causes the most rapid decrease in $|F|$ when leaving z_0 in either direction (steepest descents)
- In the limit of large t the ~~integ~~ contribution to the integral from the neighborhood of z_0 approaches the exact value of $f(t)$ asymptotically.

Neither the real nor the imaginary part of an analytic fctn can have an extremum in its region of analyticity.

Write $F(z,t) = e^{w(z,t)} = e^{u(z,t)+iv(z,t)}$

~~This is~~ $u(z,t) = \ln|F(z,t)|$ so also the absolute value cannot have an extremum.

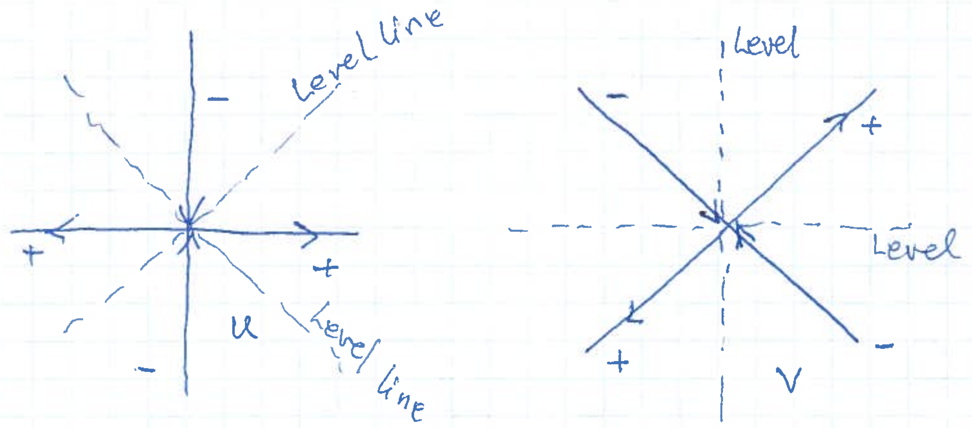
There can be a saddlepoint ($w' = 0$)

Taylor expand around z_0 $\quad w(z,t) = \underbrace{w(z_0,t)}_{w_0} + \frac{\overbrace{w''(z_0,t)}^{w_0''}}{2!}(z-z_0)^2 + \ldots$

set $w_0'' = |w_0''|e^{i\alpha}$, $z - z_0 = re^{i\vartheta}$

$$w(z,t) = w_0 + \frac{1}{2}|w_0''|\,r^2 e^{i(\alpha+2\vartheta)} + \ldots =$$

$$= w_0 + \frac{1}{2}|w_0''|\,r^2\left(\cos(\alpha+2\vartheta) + i\sin(\alpha+2\vartheta)\right) + \ldots$$



Consider $\cos(\alpha + 2\vartheta)$ maximum increase in u for $\alpha + 2\vartheta = 2n\pi$

$$\vartheta = -\frac{\alpha}{2}\ ,\ \vartheta = -\frac{\alpha}{2} + \pi$$

maximum decrease in u for $\alpha + 2\vartheta = (2n+1)\pi$

$$\vartheta = -\frac{\alpha}{2} + \left(\tfrac{1}{2}\pi \text{ or } \tfrac{3}{2}\pi\right)$$

To second order u is constant when $\alpha + 2\vartheta = (2n+1)\frac{\pi}{2}$ level line

For the imaginary part v the corresponding angles are rotated by $\frac{\pi}{4}$ making the path along maximum decrease/increase in u a level line for v, i.e. the argument is constant.

- Assume significant contributions only from a small range $0 \le r \le a$ in each of the steepest descent directions $\vartheta = -\frac{\alpha}{2} + \frac{1}{2}\pi$ and $\vartheta = -\frac{\alpha}{2} + \frac{3}{2}\pi$. We have $dz = e^{i\vartheta} dr$ along the path.

Same contribution going up to and down from the saddle point (quadratic approximation). Use expansion around z_0

$$f(t) \approx 2 e^{w_0 + i\vartheta} \int_0^a e^{-|w_0''| r^2/2} dr \qquad e^{i(\alpha + 2\vartheta)} = -1$$

~~Ass~~ If the curvature $|w_0''|$ along the steepest descent is large enough we can extend the integration to ∞ and

$$\int_0^\infty e^{-|w_0''| r^2/2} dr = \sqrt{\frac{2\pi}{|w''(z_0,t)|}}$$

and $$f(t) \approx F(z_0,t) e^{i\vartheta} \sqrt{\frac{2\pi}{|w''(z_0,t)|}}$$

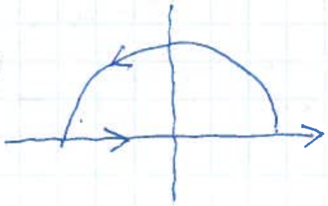
where $$\vartheta = -\frac{\arg(w''(z_0,t))}{2} + \left(\frac{\pi}{2} \text{ or } \frac{3\pi}{2}\right)$$

If $H(z,t) = g(z,t)F(z,t)$ with g varying slowly and F rapidly then $f(t) \approx g(z_0,t)\int_C F(z,t)\,dz$

Dispersion relations

The index of refraction n can have a real part determined by the phase velocity and a negative imaginary part determined by the absorption in the medium. Kronig and Kramers showed that the real part of n^2-1 can be expressed as an integral of the imaginary part and vice versa.

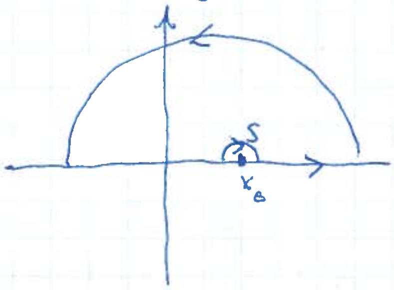
Consider $f(z)$ complex and analytic in the UHP and on the real axis. Further, $f(z)$ must go to zero on the large half-circle in the UHP so that it doesn't contribute



Then we can express $f(z_0) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(x)}{x-z_0}dx$

with z_0 in the UHP

Move z_0 onto the real axis to x_0 and avoid it through a small half-circle



No poles are contained so

$f f(x)dx$ Cauchy principal value

$$0 = \int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx + \int_S \frac{f(z)}{z-x_0}dz =$$

$$= \int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx - i\pi f(x_0)$$

So that $f(x_0) = \frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx$

Split f into real and imaginary parts $f(x_0) = u(x_0) + iv(x_0)$ ($y=0$ on real axis)

$$\Rightarrow f(x_0) = u(x_0) + iv(x_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v(x)}{x-x_0}dx - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{u(x)}{x-x_0}dx$$

So that

$$u(x_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v(x)}{x-x_0}dx$$

Dispersion relations

$$v(x_0) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x)}{x-x_0}dx$$

Suppose we have a symmetry such that $f(-x) = f^*(x)$

$$\Rightarrow \quad \underset{\text{even}}{u(-x)} + i\,\underset{\text{odd}}{v(-x)} = u(x) - i v(x)$$

Consider $v(x_0) = -\frac{1}{\pi}\int \frac{u(x)\,dx}{x-x_0} = -\frac{1}{\pi}\int_{-\infty}^{0} \frac{u(x)\,dx}{x-x_0} - \frac{1}{\pi}\int_0^{\infty}\frac{u(x)\,dx}{x-x_0}$

Set $x \to -x$ in the first integral

$$-\frac{1}{\pi}\int_{\infty}^{0} \frac{u(-x)(-dx)}{-x-x_0} = \frac{1}{\pi}\int_0^{\infty}\frac{u(x)\,dx}{x+x_0} \quad \text{and}$$

$$v(x_0) = \frac{1}{\pi}\int_0^{\infty} u(x)\left\{\frac{1}{x+x_0} - \frac{1}{x-x_0}\right\}dx = -\frac{2}{\pi}\int_0^{\infty}\frac{x_0\,u(x)\,dx}{x^2 - x_0^2}$$

Similarly for $u(x_0) = \frac{2}{\pi}\int_0^{\infty}\frac{x\,v(x)\,dx}{x^2 - x_0^2}$

Electromagnetic wave $e^{i(kx-\omega t)}$ moving along x-axis in the positive direction with velocity $v = \frac{\omega}{k}$ (ω angular frequency, k wave number)

The index of refraction is $n = ck/\omega$

and in a dielectric with conductivity σ we have

$$k^2 = \varepsilon\frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega\varepsilon}\right) \qquad \varepsilon \text{ electric permittivity and } \mu = 1$$

For $\frac{4\pi\sigma}{\omega\varepsilon} \ll 1$ (poor conductivity) we can approximate

k with its binomial expansion terminated at the second term, i.e. $k = \sqrt{\varepsilon}\,\frac{\omega}{c}\sqrt{1 + i\frac{4\pi\sigma}{\omega\varepsilon}} \simeq \frac{\omega}{c}\sqrt{\varepsilon}\left(1 + i\frac{2\pi\sigma}{\omega\varepsilon}\right) =$

$$= \sqrt{\varepsilon}\,\frac{\omega}{c} + i\frac{2\pi\sigma}{c\sqrt{\varepsilon}}$$

so $e^{i(kx-\omega t)} = e^{i\omega(x\sqrt{\varepsilon}/c - t)} \cdot e^{-2\pi\sigma x/c\sqrt{\varepsilon}}$ attenuated wave

We have $n^2 = \frac{c^2k^2}{\omega^2} = \varepsilon + i\frac{4\pi\sigma}{\omega}$ where $\varepsilon \equiv \varepsilon(\omega)$ and $\sigma \equiv \sigma(\omega)$

but $n^2 \xrightarrow[\omega\to\infty]{} 1$

consider instead n^2-1 to obtain the dispersion relation

$0 \leq \omega < \infty$

$$\mathrm{Re}\left[n^2(\omega_0)-1\right] = \frac{2}{\pi}\int_0^\infty \frac{\omega\,\mathrm{Im}\left[n^2(\omega)-1\right]d\omega}{\omega^2-\omega_0^2}$$

and $$\mathrm{Im}\left[n^2(\omega_0)-1\right] = -\frac{2}{\pi}\int_0^\infty \frac{\omega_0\,\mathrm{Re}\left[n^2(\omega)-1\right]d\omega}{\omega^2-\omega_0^2}$$

$\therefore$ A measurement of the absorption coefficient at all frequencies gives the real part of the index of refraction and vice versa