Key Points Chapter (6) 11

- Multivaluedness, branch cuts
- Cauchy - Riemann conditions
- The integral $\frac{1}{2\pi i}\oint_{|z|=r} z^n dz = \begin{cases} 0 & , n \neq -1 \\ 1 & , n = -1 \end{cases}$
- Cauchy Integral Theorem
- Cauchy Integral Formula
- Taylor expansion of complex functions
- Analytic Continuation
- Poles
- Laurent series
- Mappings

Key Points Chapter (7) 11

- Residue theorem $\oint f(z)dz = 2\pi i \sum_k \mathrm{Res}\{z_k\}$
- Trigonometric functions $z = e^{i\vartheta}$, $d\vartheta = \frac{dz}{iz}$, $\sin\vartheta = \frac{1}{2i}(z - \frac{1}{z})$
- Fourier integrals, Jordan's lemma
- Poles on the contour, Cauchy principal value
- Specific Cases (I-V)
- Finding residues
  - a) Laurent expansion
  - b) $\mathrm{Res}\{z = z_0\} = \lim_{z \to z_0} (z - z_0) f(z)$ (simple pole at z_0)
  - c) Multiple pole, order m

    $$\mathrm{Res}\{z = z_0\} = \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z - z_0)^m f(z) \right\} \cdot \frac{1}{(m-1)!}$$
  - d) $f(z) = \frac{g(z)}{h(z)}$, $h(z_0) = 0$, $g(z_0) \neq 0$, simple pole

    $$\mathrm{Res}\{z = z_0\} = \frac{g(z_0)}{h'(z_0)}$$

Chapter 11

Calculus of residues

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ be the Laurent expansion around an isolated singular point z_0. Integrate (counterclockwis along a closed contour encircling z_0. Assume no other singular point within the contour. We obtain the contributions

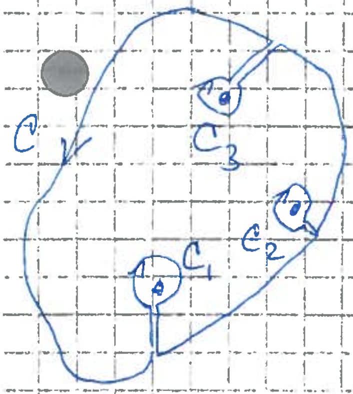
$$a_n \oint (z-z_0)^n dz = a_n \left[\frac{(z-z_0)^{n+1}}{n+1}\right]_{z_1}^{z_1} = 0 \quad \text{if } n \neq -1$$

For $n=-1$ we have $a_{-1} \oint \frac{dz}{z-z_0} = a_{-1} \int_0^{2\pi} \frac{ire^{i\vartheta} d\vartheta}{re^{i\vartheta}} = 2\pi i \, a_{-1}$

integrating along a circle around $z = z_0$.

The integral $\oint f(z) dz = 2\pi i \, a_{-1}$ and the coefficient a_{-1} of $\frac{1}{z-z_0}$ in the Laurent expansion is called the residue and gives the value.

If the contour encircles more than one singular point we add cutlines to allow each to be encircled by a small circle and get



$$0 = \oint_C f(z) dz + \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz + \oint_{C_3} f(z) dz$$

For the C_i: $\oint_{C_i} f(z) dz = -2\pi i \, a_{-1, z_i}$

where a_{-1,z_i} is the $n=-1$ coefficient in the Laurent expansion of $f(z)$ around $z = z_i$.

We get

$$\oint f(z) dz = 2\pi i \left(a_{-1,z_1} + a_{-1,z_2} + a_{-1,z_3} \right)$$

$$= 2\pi i \sum_k \mathrm{Res}(z_k)$$

Finding residues

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ be the Laurent expansion of f(

To evaluate $\oint f(z)\,dz = 2\pi i\, a_{-1}$ we only need the a_{-1} coefficient, i.e. the coefficient for z^{-1}

1) z_0 simple pole: $f(z) = \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$

a scale by $(z-z_0)$ and take the limit $z \to z_0$

$$\lim_{z\to z_0} (z-z_0) f(z) = \lim_{z\to z_0} \left\{ a_{-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+1} \right\} = a_{-1}$$

b $$f(z) = g(z)\cdot h(z) \approx \left(\frac{b_{-1}}{z} + b_0 + \dots \right)\left(\frac{c_{-1}}{z} + c_0 + \dots \right) =$$

$$= \left(\frac{b_{-1} c_{-1}}{z^2} + \frac{c_0 b_{-1} + b_0 c_{-1}}{z} + \dots \right) \to a_{-1} = c_0 b_{-1} + b$$

2) z_0 m:th order pole:

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \dots + \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Scale by $(z-z_0)^m$: $(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \dots$

$$+ a_{-1}(z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}$$

Take m-1:th derivative and let $z \to z_0$ $\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) = (m-1)!\, a_{-1}$

$$\Rightarrow \boxed{a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)}$$

3) $f(z) = \frac{g(z)}{h(z)}$, $h(z_0) = 0$, $g(z_0) \neq 0$, simple pole

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{g(z)}{\frac{h(z)}{z - z_0}} =$$

$$= \lim_{z \to z_0} \frac{g(z)}{\frac{h(z) - \overset{=0}{h(z_0)}}{z - z_0}} = \frac{g(z_0)}{h'(z_0)}$$

Specific integrals

Trigonometric functions $f(\sin\vartheta, \cos\vartheta)$ are converted t

$f(z)$ through $\sin\vartheta = \frac{1}{2i}\left(e^{i\vartheta} - e^{-i\vartheta}\right)$, $\cos\vartheta = \frac{1}{2}\left(e^{i\vartheta} + e^{-i\vartheta}\right)$

and $z = e^{i\vartheta}$, $dz = ie^{i\vartheta}d\vartheta$ so that

unit circle

$$d\vartheta = -i\frac{dz}{e^{i\vartheta}} = -i\frac{dz}{z}, \quad \sin\vartheta = \frac{1}{2i}\left(z - z^{-1}\right)$$

$$\cos\vartheta = \frac{1}{2}\left(z + z^{-1}\right)$$

$$\int_0^{2\pi} f(\sin\vartheta, \cos\vartheta)\,d\vartheta = -i\oint f\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right)\frac{dz}{z} =$$

$$= (-i)\,2\pi i\,\Sigma_i\{\text{residues within the unit circle}\}$$

i.e. residues with respect to $\frac{f(z)}{z}$

Ex. $\quad I = \displaystyle\int_0^{2\pi}\frac{d\vartheta}{1+\varepsilon\cos\vartheta} \qquad |\varepsilon| < 1$

$$\cos\vartheta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad d\vartheta = -i\frac{dz}{z}$$

and $\displaystyle I = -i\oint_{\text{unit circle}}\frac{dz}{z\{1+\frac{\varepsilon}{2}(z+\frac{1}{z})\}} = -i\frac{2}{\varepsilon}\oint\frac{dz}{z^2+\frac{2}{\varepsilon}z+1}$

Poles: $z^2 + \frac{2}{\varepsilon}z + 1 = 0 \Rightarrow z = -\frac{1}{\varepsilon} \pm \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2} \equiv z_\pm$

$|z_+| < 1$ but $|z_-| > 1$ gives ⊛

$$I = -\frac{i2}{\varepsilon}(2\pi i)\,\text{Res}\,(z = z_+) = \frac{4\pi}{\varepsilon}\lim_{z\to z_+}\left(z + \frac{1}{\varepsilon} - \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}\right)f(z$$

$$= \frac{4\pi}{\varepsilon}\lim_{z\to z_+}\frac{1}{z+\frac{1}{\varepsilon}+\frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}} = \frac{4\pi}{\varepsilon}\frac{1}{-\frac{1}{\varepsilon}+\frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}+\frac{1}{\varepsilon}+\frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}}$$

$$= \frac{2\pi}{\sqrt{1-\varepsilon^2}}$$

⊛ Note that $z_+ \cdot z_- =$

$$= \left(-\frac{1}{\varepsilon}\right)^2 - \left[\frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}\right]^2 = 1$$

Specific integrals $\int_{-\infty}^{\infty} f(x)\,dx$

(1) Assume $f(z)$ analytic in the upper half plane excep
(UHP)
at a finite number of poles

(2) $f(z)$ goes to zero as ~~$\frac{1}{z^2}$~~ when $z \to \infty$ for $0 \leq \arg(z) \leq \pi$

Faster than 1/|z|

If (1) and (2) apply we can extend the integratio to a closed path which contains $+R \leq x \leq R$ along the real axis and the half circle from $+R$ to $-R$ where the integral along the half circle in the limi of $R \to \infty$ gives a zero contribution, i.e.

$$\oint f(z)\,dz = \lim_{R\to\infty} \int_{+R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{0}^{\pi} f(Re^{i\vartheta})\, iRe^{i\vartheta} d\vartheta$$

$$= 2\pi i \sum_{UHP} \{\text{contained residues}\}$$

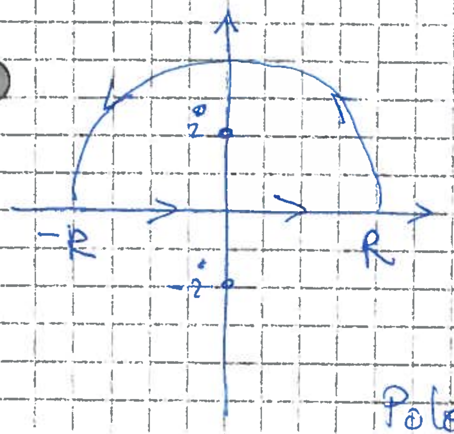
According to (2) the second integral is zero and

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{UHP} \{\text{contained residues}\}$$

Ex.

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

We close the contour in the UHP and get



$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2} + \underbrace{\lim_{R\to\infty} \int_{0}^{\pi} \frac{iRe^{i\vartheta} d\vartheta}{1+R^2 e^{2i\vartheta}}}_{\to 0} = 2\pi i \sum_{\text{contained}} \text{Res}\{z_i\}$$

$$\Rightarrow \int_{+\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \sum_{\text{contained}} \text{Res}\{z_i\}$$

Poles from $1+x^2 = 0 \to x = \pm i$

Only the pole at $z = i$ is contained within the region

Expand $f(z)$ as $f(z) = \frac{1}{z^2+1} = \frac{a_{-1}}{z-i} + a_0 + \sum_{n=1}^{\infty} a_n (z-i)^n$

a_{-1} is obtained directly from $\lim_{z\to i} (z-i) f(z) = \lim_{z\to i} \frac{(z-i)}{(z-i)(z+i)} = \frac{1}{2i}$

simple pole!

$$\boxed{\Rightarrow I = 2\pi i \cdot \frac{1}{2i} = \pi}$$

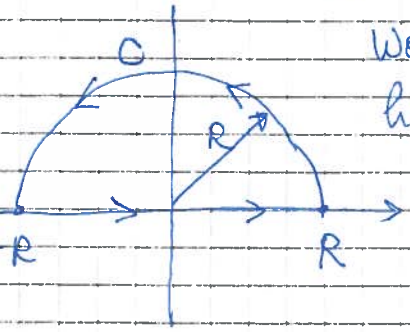
Fourier integrals

$$\int_{-\infty}^{\infty} f(x)e^{iax}\,dx \qquad a\in R, \quad a>0$$

(1) $f(z)$ analytic in UHP except for a finite number of poles

(2) $\lim_{|z|\to\infty} f(z) = 0, \quad 0 \le \arg(z) \le \pi$

As before we have that $\oint f(z)e^{iaz}dz = 2\pi i \sum \{\text{contained residues}\}$



We have to show that the integral over the half circle ~~goes~~ is zero, i.e. $\lim_{R\to\infty} \int_C f(z)e^{iaz}dz = 0$

$$I_R = \int_0^{\pi} f(Re^{i\vartheta})e^{iaR\cos\vartheta - aR\sin\vartheta}\, iRe^{i\vartheta}d\vartheta$$

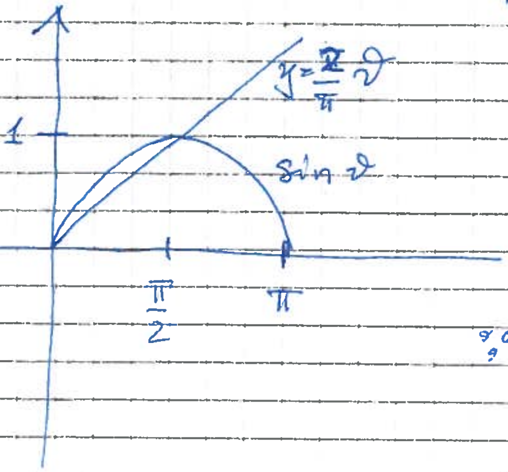
for R large enough that $|f(z)| = |f(Re^{i\vartheta})| < \varepsilon$ we have that $|I_R| \le \varepsilon R \int_0^{\pi} e^{-aR\sin\vartheta}d\vartheta = 2\varepsilon R\int_0^{\pi/2} e^{-aR\sin\vartheta}d\vartheta$

Note 11.3.2

In the interval $[0, \frac{\pi}{2}]$ we have $\frac{2}{\pi}\vartheta \le \sin\vartheta$ so that

$$I_R \le 2\varepsilon R\int_0^{\pi/2} e^{-2aR\vartheta/\pi}d\vartheta = 2\varepsilon R\left[\frac{e^{-2aR\vartheta/\pi}}{-2aR/\pi}\right]_0^{\pi/2} =$$

$$= \frac{2\varepsilon R}{2aR/\pi}\left\{1 - e^{-aR}\right\} \Rightarrow \lim_{R\to\infty}|I_R| \le \frac{\pi\varepsilon}{a}$$



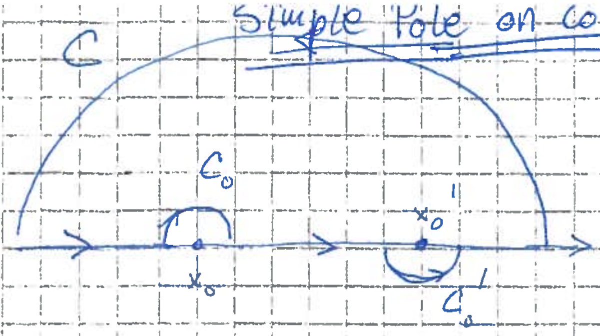
Let $\varepsilon \to 0$ and $R\to\infty \Rightarrow I_R \to 0$

$$\therefore \int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i\sum_{} \text{Residues in UHP} \quad a>$$

Jordan's Lemma

$a<0$ same thing in LHP

Simple Pole on contour



$$\int_{C_0} \frac{dz}{z-x_0} = i\int_{\pi}^{0} \frac{\delta e^{i\vartheta}\,d\vartheta}{\delta e^{i\vartheta}} = -i\pi \quad (-i\pi a_{-1})$$

$$\int_{C_0'} \frac{dz}{z-x_0'} = i\int_{\pi}^{2\pi} \frac{\delta e^{i\vartheta}\,d\vartheta}{\delta e^{i\vartheta}} \to i\pi \quad (i\pi a_{-1}')$$

According to the residue theorem we have

$$\oint f(z)\,dz = \int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{C_0} f(z)\,dz + \int_{x_0+\delta}^{x_0'-\delta} f(x)\,dx + \int_{C_0'} f(z)\,dz +$$

$$+ \int_{x_0'+\delta'}^{\infty} f(x)\,dx + \int_{C} f(z)\,dz = 2\pi i \sum \{\text{contained residues}\}$$

Simple poles

x_0 excluded $\Rightarrow$ contributes $-i\pi a_{-1}$ on the left hand side

x_0' included $\Rightarrow$ contributes $i\pi a_{-1}'$ on the left and $2i\pi a_{-1}'$ on th right hand side

If we take the contour C such that $\int_C f(z)\,dz \to 0 \Rightarrow$

$$\int_{-\infty}^{x_0-\delta} f(x)\,dx - i\pi a_{-1} + \int_{x_0+\delta}^{x_0'-\delta'} f(x)\,dx + i\pi a_{-1}' + \int_{x_0'+\delta'}^{\infty} f(x)\,dx = 2\pi i a_{-1}'$$

The contribution is the same irrespective of whether the pole is included or excluded

$$I(\delta,\delta') \equiv \int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta}^{x_0'+\delta} f(x)\,dx + \int_{x_0'+\delta}^{\infty} f(x)\,dx = 2\pi i a_{-1}' - \pi i a_{-1}' + \pi i a_{-1} =$$

$$= \pi i (a_{-1} + a_{-1}')$$

Let $\delta, \delta' \to 0$ and define

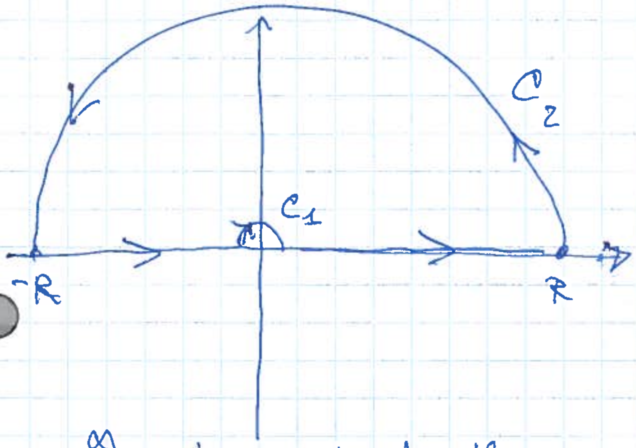
$$\lim_{\delta,\delta' \to 0} I(\delta,\delta') = P\int_{-\infty}^{\infty} f(x)\,dx \quad \text{Cauchy principal value}$$

The integral becomes well-behaved and defined through the ~~[illegible]~~ cancellation of contributions in the vicinity of the singularity (simple pole), where $f(x) \approx \frac{a_{-1}}{x-x_0}$ is odd with respect to x. Taking the symmetric limit allows for a cancellation of contribution.

The limit $\lim_{\delta \to 0} \left\{ \int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta}^{\infty} f(x)\,dx \right\}$ is the Cauchy principal value, if it exists

Ex. 11.8.5 $\quad I = \int_0^\infty \frac{\sin x}{x} dx$

The integral can be viewed as the imaginary part of $I_2 = \frac{1}{2} \fint_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$ with simple pole at $x = 0$ and residue $a_{-1} = 1$



Use Jordan's lemma in the UHP and avoid the pole on the real axis

$$\oint \frac{e^{iz}}{z} dz = -\int_{-R}^{-\delta} \frac{e^{ix}}{x} dx + \int_{C_1} \frac{e^{iz}}{z} dz + \int_{\delta}^{R} \frac{e^{ix}}{x} dx +$$

$$+ \underbrace{\int_{C_2} \frac{e^{iz}}{z} dz}_{= 0 \text{ (Jordan's lemma)}} = 0$$

$\fint_{-\infty}^{\infty} \frac{e^{iz}}{z} dz$ denotes the Cauchy principal value

$$\oint \frac{e^{iz}}{z} dz = \int_{C_1} \frac{e^{iz}}{z} dz + \fint_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = 0$$

$$\lim_{\delta \to 0} \int_{C_1} f(z) dz = \lim_{\delta \to 0} \sum_n a_n \int_{C_1} z^n dz = \lim_{\delta \to 0} \sum i \delta^{n+1} a_n \int_{\pi}^{0} e^{i(n+1)} d\vartheta = -i a_{-1} \pi$$

$$\therefore \fint_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i a_{-1} = \pi i$$

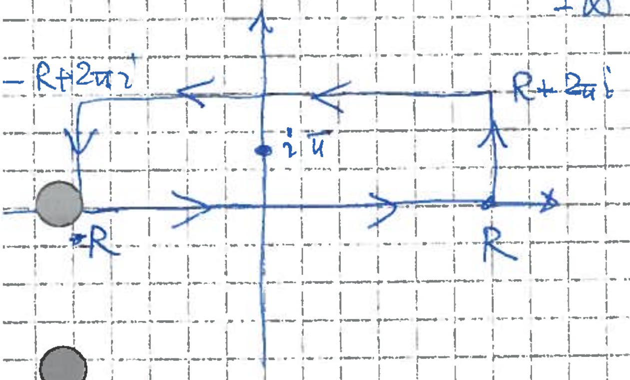
Imaginary part and scaling by $\frac{1}{2}$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

~~Instead doing the substitution $\omega \to \omega - i\delta$ would have~~
~~given the incoming wave $\sim e^{i\delta}$~~

Ex. Evaluate the definite integral

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}dx, \quad 0<a<1$$



Make a closed path according to the figure.

$$\oint \frac{e^{az}}{1+e^{z}}dz = \lim_{R\to\infty}\left\{ \int_{-R}^{R} \frac{e^{ax}}{1+e^{x}}dx - e^{i2\pi a}\int_{-R}^{R}\frac{e^{ax}}{1+e^{x}}dx \right\} = \left(1-e^{i2\pi a}\right)\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}dx$$

The vertical segments do not contribute when $R \to \pm\infty$

We have poles when $e^{z} = e^{x}\cdot e^{iy} = -1$, i.e. $z = i\pi$

Do a Taylor expansion around $z = i\pi$

$$1+e^{z} = 1+e^{z-i\pi}\cdot e^{i\pi} = 1-e^{z-i\pi} =$$

$$= -(z-i\pi)\left(1+\frac{z-i\pi}{2!}+\frac{(z-i\pi)^2}{3!}+\dots\right)$$

This is enough to get the residue:

$$\lim_{z\to i\pi}\frac{(z-i\pi)e^{az}}{1+e^{z}} = \lim_{z\to i\pi}\frac{(z-i\pi)e^{az}}{-(z-i\pi)\left(1+\frac{z-i\pi}{2!}+\dots\right)} = -e^{+i\pi a}$$

Thus $$\oint \frac{e^{az}}{1+e^{z}}dz = 2\pi i\left(-e^{+i\pi a}\right) = \left(1-e^{i2\pi a}\right)\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}dx = \frac{-2\pi i\, e^{i\pi a}}{e^{i\pi a}\left(e^{-i\pi a}+e^{i\pi a}\right)} = \frac{\pi}{\sin \pi a}$$

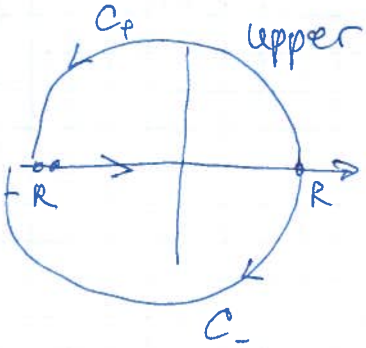
Complex integration – calculus of residues

General problem $I = \int_{-\infty}^{\infty} f(x)dx$; open curve, but the integra theorem is valid for close contours.

Let $f(z)$ be the analytic continuation of $f(x)$

For the first four cases we assume $f(z)$ single-valued, i.e. no need to introduce a cutline to avoid branch points.

Case I: If $|zf(z)| \to 0$ when $|z| \to \infty$ then the curve can be closed using a semicircle in either the upper or lower half-plane.



The integral becomes $I = \pm 2\pi i \sum_{\text{enclosed}} \text{Res}\{f(z)\} - I$

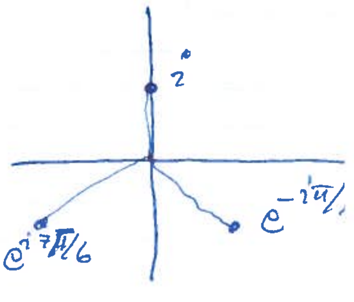
where $I_R \equiv I(\text{semicircle}) =$

$$= \lim_{R\to\infty} \int_{\vartheta=0}^{\vartheta=\pm\pi} f(Re^{i\vartheta})\, iRe^{i\vartheta} d\vartheta = \lim_{|z|\to 0} i\int_{\vartheta=0}^{\vartheta=\pm\pi} z f(z) d\vartheta \leq$$

$$\leq \pm\pi i \lim_{R\to\infty} \text{Max}\left[z f(z)\right] = 0$$

$$\Rightarrow I = \pm 2\pi i \sum_{\text{enclosed}} \text{Res}\{f(z)\}$$

Example: $I = \int_{-\infty}^{\infty} \frac{dx}{x^3+i}$ $\quad \lim_{|z|\to\infty} \left|\frac{z}{z^3+i}\right| = 0$



Three poles: $z = i,\ e^{-i\pi/6},\ e^{i7\pi/6}$

Close in upper half plane:

$$I = \oint_{C_+} \frac{dz}{z^3+i} = 2\pi i\, \text{Res}\{f(z=i)\} = 2\pi i \cdot \left(\frac{1}{3z^2}\right)_{z=i} = -\frac{2\pi}{3} i$$

$$\left[\text{Residue from } \text{Res}\left[\frac{P(z)}{Q(z)}\right]_{z=i} = \left(\frac{P(z=i)}{\frac{d}{dz}Q(z)\big|_{z=i}}\right)\right.$$

Close in lower half plane: $I = \oint_{C_-} \frac{dz}{z^3+i} = -2\pi i \left\{\text{Res}\left[f(z=e^{-i\pi/6})\right] + \text{Res}\left[f(z=e^{i7\pi/6}\right.\right.$

$$= -\frac{2\pi i}{3}\left\{e^{i\pi/3} + e^{-\frac{7\pi i}{3}}\right\} = -\frac{2\pi i}{3}\left\{e^{i\pi/3} + e^{-i\pi/3}\right\} = -\frac{2\pi i}{3} \cdot 2\cos\frac{\pi}{3} = -\frac{2\pi i}{3}$$

Case II: Application of Jordan's Lemma: $I = \int_{-\infty} g(x) e^{i\lambda x} dx$, $\lambda \in R$

If $\lim_{|z| \to \infty} g(z) = 0$ then the path can be closed using a non-contributing half-circle in the upper half plane ($\lambda > 0$) or the lower half plane ($\lambda < 0$).
The contribution from the half-circle is zero and

$$I = \pm 2\pi i \sum_{\text{enclosed}} \text{Res}\{ g(z) e^{i\lambda z} \}$$

Example: $I = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x + ia} dx$, $\lambda, a \in R, > 0$

case 1: $I = \oint \frac{e^{i\lambda z}}{z + ia} dz = 0 \Rightarrow \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x + ia} dx = 0$

The pole is in the lower half plane

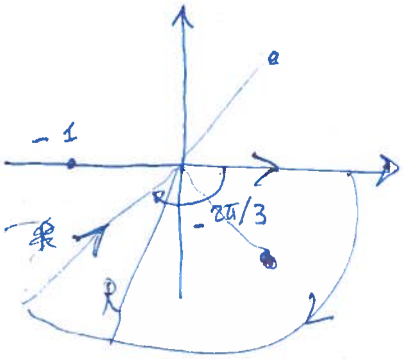
case 2: $I = \oint_{LHP} \frac{e^{-i\lambda z}}{z + ia} dz = -2\pi i \text{Res}\{f(z = -ia)\} = -2\pi i e^{-i\lambda(-ia)} =$
$= -2\pi i e^{-a\lambda}$

Put together: $\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{i\lambda x} - e^{-i\lambda x}}{x + ia} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x + ia} dx = \frac{1}{2i}\left(0 + 2\pi i e^{-a\lambda}\right) =$
$= \pi e^{-a\lambda}$

Special case: $\int_0^{\infty} \frac{\sin kx}{x} dx = \lim_{a \to 0} \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin kx}{x + ia} dx = \frac{\pi}{2}$

Case III: If $f(z)$ along a different path is proportional to $f(z)$ along the given/desired path then the two can often be combined to a closed contour.

Example: $I = \int_0^{\infty} \frac{dx}{1 + x^3}$, $z^3 = r^3$ along the real axis but also for $z = r e^{i2\pi/3}$ and $z = r e^{-i2\pi/3}$



Close with arch segment between $\vartheta = 0$ and $\vartheta = -2\pi/3$. The contribution from the arch segment vanishes for $R \to \infty$, we get

$$I = \oint \frac{dz}{1 + z^3} = I + I'$$

The contribution, I', along the radius $re^{-i2\pi/3}$ is

$$I' = \int_{\infty}^{0} \frac{e^{-i2\pi/3}\,dr}{1+r^3} = -e^{-i2\pi/3} \cdot I$$

The pole at $z=e^{-i\pi/3}$ is enclosed. Evaluate the residue as

$$\mathrm{Res}\{F(z)\}_{z=z_0} = \lim_{z\to z_0}\frac{g(z)}{h(z)}(z-z_0) = \lim_{z\to z_0}\frac{g(z)}{\frac{h(z)-h(z_0)}{z-z_0}} = \frac{g(z_0)}{h'(z_0)}$$

where $h(z_0)=0$, $g(z_0)\neq 0$ and $\frac{1}{h(z)}$ has a simple pole at z_0

$$\left.\begin{array}{l} g(z)=1 \\ h(z)= 1+z^3 \end{array}\right\} \Rightarrow \mathrm{Res}\{z=e^{-i\pi/3}\}\frac{1}{1+z^3} = \frac{1}{3e^{-i2\pi/3}}$$

$$\therefore\ (1-e^{-i2\pi/3})I = \oint \frac{dz}{1+z^3} = -\frac{2\pi i}{3e^{-i2\pi/3}} \qquad \text{(path is clockwise)}$$

$$\Rightarrow I = -\frac{2\pi i}{3}\cdot\frac{1}{e^{-i2\pi/3}-e^{-i4\pi/3}} = \frac{2\pi i}{3}\cdot\frac{1}{e^{i2\pi/3}-e^{-i2\pi/3}} = \frac{\pi}{3\sin(\frac{2\pi}{3})} = \frac{2\pi}{3\sqrt{3}}$$

Case IV: Angular integrations $I=\int_0^{2\pi} G(\sin\vartheta,\cos\vartheta)\,d\vartheta$

Change of variable $z=e^{i\vartheta}$, $dz=ie^{i\vartheta}d\vartheta$

$$G(\sin\vartheta,\cos\vartheta) = G\left(\frac{1}{2i}\left(z-\frac{1}{z}\right),\frac{1}{2}\left(z+\frac{1}{z}\right)\right) = f(z) \Rightarrow I=\oint f(z)\frac{dz}{iz}$$

Example: $I=\int_0^{2\pi}\frac{d\vartheta}{a+b\sin\vartheta}$, $a>|b|>0$

$$I=\oint\frac{dz}{iz}\,\frac{1}{a+b(\frac{1}{2i})(z-\frac{1}{z})} = \frac{2}{b}\oint\frac{dz}{(z^2-1)+2i(\frac{a}{b})z}$$

The poles of the integrand are at $z_\pm = \left\{-\frac{a}{b}\pm\left(\frac{a^2}{b^2}-1\right)^{1/2}\right\}i$

since $z_+\cdot z_-=1$ one of the poles is inside the unit circle while the other is outside.

If $b>0$ then $\frac{a}{b}>1$ and $-iz_- = -\frac{a}{b}-\left(\frac{a^2}{b^2}-1\right)^{1/2} < -1$ and z_- is outside the unit circle while z_+ is inside.

If $b<0$ then instead z_- is inside. In either case

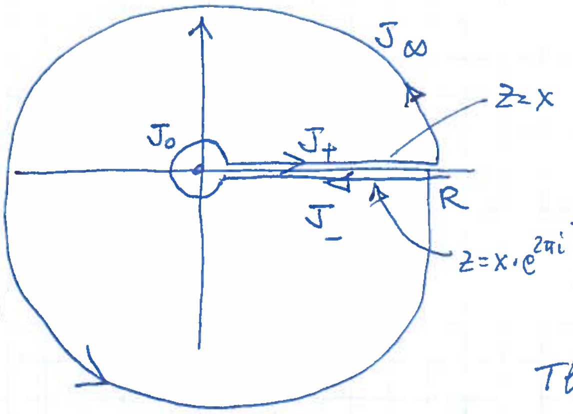
$$I=\frac{2}{b}\,2\pi i\,\mathrm{Res}[f(z_{inside})] = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Case 5: $f(z)$ not single-valued when going around pole

$$I = \int_0^\infty f(x)\,dx \quad , \quad \text{pole at } z=0 \text{ and } z=\infty$$

Introduce branch cut along $x=[0,\infty]$ and consider the integral

$$J = \oint f(z) \ln z \, dz =$$



$$= J_+ + J_- + J_0 + J_\infty$$

Required: $|z f(z) \ln z| \to 0$ for $|z| \to 0, \infty$

Then only the paths above and below the branch cut will contribute.

$$J_+ = \int_0^\infty f(x) \ln x \, dx \; ; \quad J_- = \int_\infty^0 f(x)(\ln x + 2\pi i)\,dx$$

The scaling by the logarithm cancels

$$J = J_+ + J_- = \int_0^\infty f(x) \ln x\, dx - \int_0^\infty f(x)(\ln x + 2\pi i)\,dx = -2\pi i\, I$$

and $I = -\sum_{\text{enclosed}} \text{Res}\{f(z) \ln z\} \cdot 2\pi i$

Finding the residue, i.e. a_{-1} in $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

1) Simple pole at z_0: scale by $z-z_0$ and take the limit of $z \to z_0$

i.e. $f(z) = a_{-1} \cdot \frac{1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$\Rightarrow a_{-1} = \lim_{z \to z_0} (z-z_0) f(z)$$

2) multiple pole, order m: $f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n$

a) scale by $(z-z_0)^m$

$$\Rightarrow (z-z_0)^m f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^{n+m} = a_{-m} + \ldots + a_{-1}(z-z_0)^{m-1} + a_0(z$$

take derivative $\frac{d^{m-1}}{dz^{m-1}} \left\{(z-z_0)^m f(z)\right\} = (m-1)!\, a_{-1} + \frac{m!}{(m-1)} a_0 (z-z_0)^{*} + \ldots$

divide by $(m-1)!$ and take limit $z \to z_0 \Rightarrow a_{-1}$

3) $f(z) = \dfrac{g(z)}{h(z)}$ where $h(z_0) = 0$, $g(z_0) \neq 0$

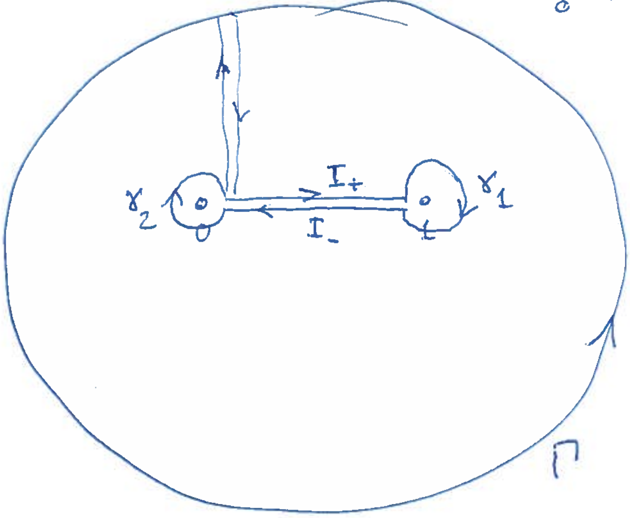
simple pole

$$a_{-1} = \lim_{z \to z_0} (z-z_0) f(z) = \lim_{z \to z_0} \frac{(z-z_0)\, g(z)}{h(z)} =$$

$$= \lim_{z \to z_0} g(z) \cdot \frac{1}{\dfrac{h(z)-h(z_0)}{z-z_0}} = \frac{g(z_0)}{h'(z_0)}$$

11.8.27   Show that $\int_0^1 \frac{1}{(x^2-x^3)^{1/3}} dx = \frac{2\pi}{\sqrt{3}}$

Multivaluedness



$$\oint \frac{dz}{(z^2-z^3)^{1/3}} = \oint \frac{dz}{z^{2/3}(1-z)^{1/3}}$$

at 1

singular and branchpoint at 0

Singly connected

Take a branch for I_+ which is real and positive above the cut so that $I_+ = \int_0^1 \frac{dx}{(x^2-x^3)^{1/3}}$

The small circles γ_1 and γ_2 do not contribute since for γ_1 the singular factor goes as ~~z~~ $r^{-1/3}$ and for γ_2 as $r^{-2/3}$ while $dz = ire^{i\vartheta}d\vartheta$

On I_- below the cut we still have $z^{2/3} = x^{2/3}$ but instead of $(1-x)^{1/3}$ we have $e^{-i2\pi/3}(1-x)^{1/3}$ (going clockwise)

The integrand for I_- becomes $e^{2\pi i/3}/(x^2-x^3)^{1/3}$ and $I_- = - e^{2\pi i/3} \cdot I_+$ (integration from 1 to 0)

On the large circle the integrand becomes $1/(-1)^{1/3} z$ and we must select the proper branch for $(-1)^{1/3}$, i.e select n in $(-1)^{1/3} = (e^{i(2n+1)\pi})^{1/3} = e^{i(2n+1)\pi/3} = \{e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}, \ldots\}$

Consider large, real positive z ; $z^{2/3}$ remains $x^{2/3}$ but $(1-x)^{1/3}$ becomes $|1-x|^{1/3} e^{-i\pi/3}$ (angle π clockwise)

Asymptotically then we get $1/(e^{-i\pi/3} \cdot z)$. The integral over $\frac{1}{z}$ around any circle is $2\pi i$ so from Γ we get $2\pi i\, e^{i\pi/3}$

Putting it all together we have

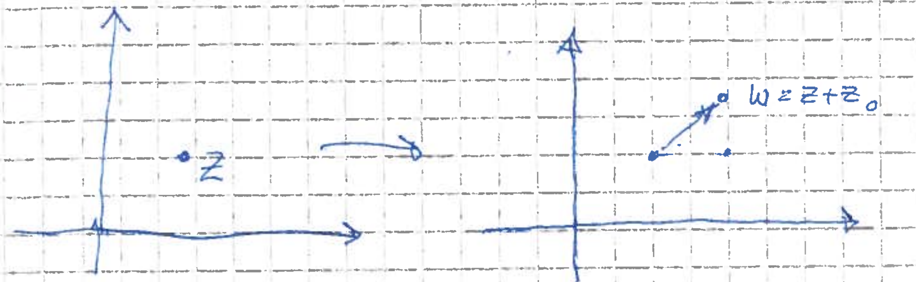
$$0 = \oint \frac{dz}{z^{2/3}(1-z)^{1/3}} = I - e^{2\pi i/3} I + 2\pi i\, e^{\pi i/3}$$

$$I(1-e^{2\pi i/3}) = -2\pi i\, e^{\pi i/3}$$

$$I = -2\pi i \frac{e^{\pi i/3}}{1-e^{2\pi i/3}} = -\frac{2\pi i}{e^{-i\pi/3} - e^{\pi i/3}} = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

The function $w=f(z)=u(x,y)+iv(x,y)$ gives a mapping of the complex plane

Ex. $w=z+z_0$ gives a translation



$w=z\cdot z_0$ gives an elongation (contraction) and rotation

$z=re^{i\vartheta}$, $z_0=\varrho e^{i\varphi}$ $\Rightarrow$ $w=r\varrho e^{i(\vartheta+\varphi)}$

Inversion: $w=\frac{1}{z}$ $\qquad w=\varrho e^{i\varphi}=\frac{1}{r}e^{-i\vartheta}$ $\Rightarrow$ $\varrho=\frac{1}{r}$, $\varphi=-\vartheta$

How are contours transformed?

$$w=u+iv=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$$

$$\Rightarrow u=\frac{x}{x^2+y^2} \qquad x=\frac{u}{u^2+v^2}$$

$$v=-\frac{y}{x^2+y^2} \qquad y=-\frac{v}{u^2+v^2}$$

A circle $x^2+y^2=r^2$ $\Rightarrow$ $\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}=\frac{1}{u^2+v^2}=r^2$

$\therefore$ a new circle $u^2+v^2=\frac{1}{r^2}=\varrho^2$, radius $\frac{1}{r}$

The horizontal line $y=C_1$ is transformed to

$-\frac{v}{u^2+v^2}=C_1$ $\Rightarrow$ $u^2+\left(v+\frac{1}{2C_1}\right)^2=\frac{1}{(2C_1)^2}$, i.e. a circle with radius $\frac{1}{2C_1}$ at $\left(0,-\frac{1}{2C_1}\right)$

Take the transformation/function/mapping

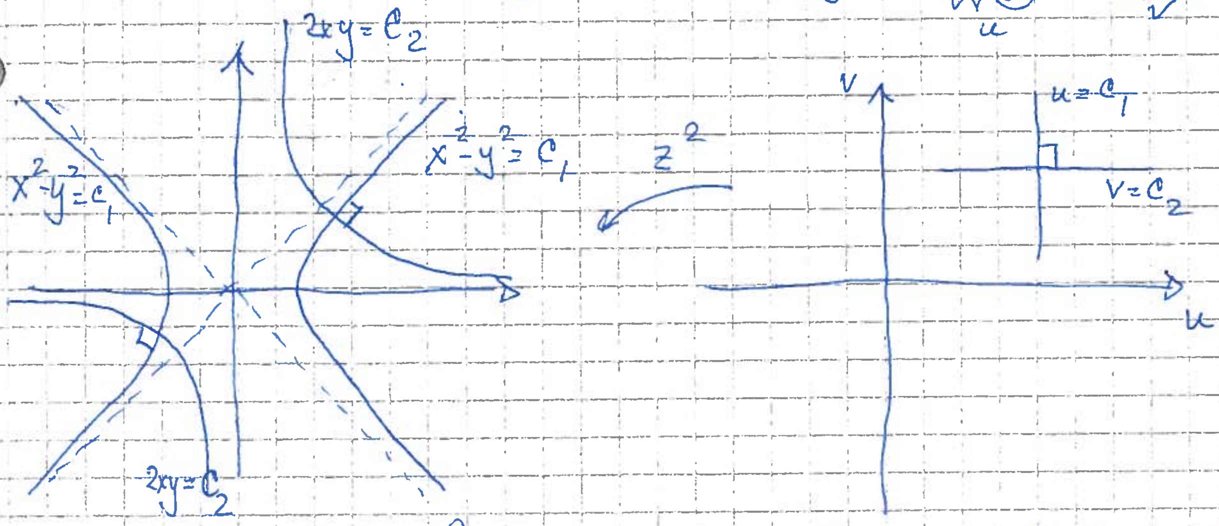
$f(z) = z^2 = \underbrace{\varrho e^{i\varphi}}_{\text{new point}}$. where for $z = re^{i\vartheta}$ $\varrho = r^2$, $\varphi = 2\vartheta$

The mapping maps the upper half plane onto the entire complex plane and also the lower half plane ~~is~~ is expanded to cover the full complex plane. The function thus maps two points onto the same point z (cf $x^2 = 1$ ~~folding~~ taking the two points $x = \pm 1$ onto the same point and actually folding the negative real axis onto the positive)

∴ The points $re^{i\varphi}$ and $re^{i(\varphi+\pi)}$ are both mapped onto $r^2 e^{i2\varphi}$. Special case $\varphi = 0$ $re^{i\pi} = -r$ and $re^{0} = r$ both become r^2

Since $f(z) = z^2$ is analytical the lines $u = C_1$ and $v = C_2$ in the w-plane, $w(x,y) = u(x,y) + iv(x,y)$, will be orthogonal where they meet. This gives an easy way to analyse coordinate systems.

$$f(z) = z^2 = (x+iy)^2 = \underbrace{x^2-y^2}_{u} + i\,\underbrace{2xy}_{v}$$



New hyperbolical locally orthogonal coordinate system

Chap. 6.8 (optional) shows that analytic mappings are conformal, ie preserve angles

Riemann surface

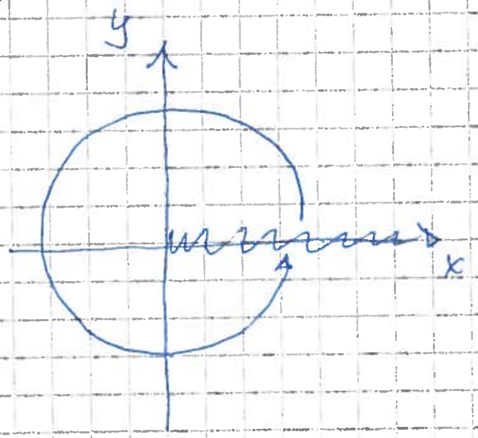
Transformation $w(z) = z^{1/2}$

with $z = re^{i\vartheta}$ we get $w(z) = \rho e^{i\varphi} = r^{1/2} e^{i\vartheta/2} \Rightarrow \rho = \sqrt{r}$

$\varphi = \vartheta/2$

Now we must instead make two full circles in order to map the full ~~plane~~ $w(z)$ plane and the same point $re^{i\vartheta}$ and $re^{i(\vartheta+2\pi)}$ in the z-plane is mapped onto two different points in the $w(z)$ plane.

Vi can recast $w(z)$ as a single valued function through a cut line limiting the argument to $0 \le \vartheta < 2\pi$. The cutline ~~goes through~~ joins the branch points $z=0$ and $z=\infty$



~~By constructing~~ At the cutline we add a second complex plane taking the argument upp to 4π and then rejoining the first plane at that point This is called a Riemann surface