Mathematical Methods for Physicists, FK7048, fall 2018
Exercise sheet 8, **due monday october 29th, 10:00**

This is a hard deadline since we want to have the possibility to discuss hand-in problems at the lecture. We will not accept hand-ins after this deadline. Email your nice solutions to Eddy (ardonne@fysik.su.se). Remember to write your email address on the top of the first page of your solutions! Scanned solutions will be accepted if, and only if, the quality is good enough to be read and corrected. Please submit solutions in electronic form in **one PDF file** only.
Total amount of points: 13 p + 1 bp.

1 The Basel problem

The purpose of this exercise is to calculate the sum of the following (converging) series by different methods:

$$S = \sum_{n=1}^{+\infty} \frac{1}{n^2} \tag{1}$$

(a) (1p) As seen in class, apply the residue theorem toan appropriate function with an appropriate contour.

You can construct a 2π-periodic function f over $\mathbb{R}$ by defining it on a period, for instance the interval $]-\pi,\pi]$. Then you are allowed to expand f in Fourier series.

(b) (1p) Take $f(x) = x$ on $]-\pi,\pi]$ and apply Parseval's theorem to find S.

(c) (1p) Take $f(x) = x^2$ on $]-\pi,\pi]$ and apply Dirichlet's theorem in a particular point to get S.

2 Retarded potentials

Let $\Box = \frac{1}{c^2}\partial_t^2 - \Delta$ be the D'Alembertian in 3D, $V(\boldsymbol{r},t)$ the electric potential, $\boldsymbol{A}(\boldsymbol{r},t)$ the magnetic vector potential, $\rho(\boldsymbol{r},t)$ the charge density and $\boldsymbol{j}(\boldsymbol{r},t)$ the electric current density. One can show that Maxwell's equations in the Lorenz[1]gauge are equivalent to wave equations with source terms:

$$\frac{1}{c^2}\frac{\partial V}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \tag{2}$$

$$\Box\, \boldsymbol{A} = \mu_0 \boldsymbol{j} \tag{3}$$

$$\Box\, V = \frac{\rho}{\epsilon_0} \tag{4}$$

(a) (2p) Calculate the retarded Green's function $G(\boldsymbol{r},t;\boldsymbol{r'},t')$ of the D'Alembertian using a space-time Fourier transform defined as follows for functions[2] $f \in L^2(\mathbb{R}^4)$:

$$\widehat{f}(\boldsymbol{k},\omega) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} f(\boldsymbol{r},t)\; e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\; d^3r\; dt \tag{5}$$

$$f(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \widehat{f}(\boldsymbol{k},\omega)\; e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\; d^3k\; d\omega \tag{6}$$

Hints: - Actually $G(\boldsymbol{r},t;\boldsymbol{r'},t') = G(\boldsymbol{r}-\boldsymbol{r'},t-t')$, moreover one can take $\boldsymbol{r'} = \boldsymbol{0}$ and $t' = 0$ during the calculation. Justify this.
- When inverting the Fourier transform in time, choose your contour so that $G(\boldsymbol{r},t) = 0$ for $t \leq 0$. Justify this choice.

[1]Note that the name is spelled correctly, this gauge is due to Lorenz, the transformations are due to Lorentz.
[2]Fourier transformation is an isomorphism only on Schwartz spaces and by a subtle extension to L^2 spaces, however the Fourier inversion formula also holds if the function and its Fourier transform both belong to an L^1 space. In a nutshell, be always careful when inverting a Fourier transform!

(b) (0.5p) Infer the so-called retarded potentials:

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3} \frac{\rho\left(\boldsymbol{r'}, t - \frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r'}|}\, d^3r' \tag{7}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{\boldsymbol{j}\left(\boldsymbol{r'}, t - \frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r'}|}\, d^3r' \tag{8}$$

3 Laplace transform of the sine cardinal

Let $\mathcal{L}$ be the Laplace transformation.

(a) (1p) Show that:

$$\mathcal{L}\left(\frac{\sinh(at)}{at}\right)(x) = \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) \tag{9}$$

Hint: You know $\mathcal{L}\left(\frac{\sinh(at)}{a}\right)(x) = \frac{1}{s^2-a^2}$.

(b) (0.5p) Use the result of the previous question to deduce that:

$$\mathcal{L}\left(\frac{\sin(at)}{at}\right)(x) = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) \tag{10}$$

4 On the Schrödinger equation

The three-dimensional, time-independent Schrödinger equation can, after a rescaling of the energy $E = \frac{\hbar^2}{2m}\epsilon$ and potential $V = \frac{\hbar^2}{2m}U$, be written as:

$$-\Delta\psi(x,y,z) + U(x,y,z)\ \psi(x,y,z) = \epsilon\ \psi(x,y,z) \tag{11}$$

where Δ is the Laplacian. For central potentials, it is interesting to use spherical coordinates. Then, one can show that solutions are of the form $\psi(r,\theta,\varphi) = R(r)\ Y_{lm}(\theta,\varphi)$ where Y_{lm} are the spherical harmonics and the radial function satisfies the following 2nd order linear ODE:

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\frac{l(l+1)}{r^2} + U(r)\right)R(r) = \epsilon\ R(r) \tag{12}$$

(a) (1p) The free particle has a constant potential energy which one can always take equal to zero. For $U = 0$, show that the only physical solution of (12) is $R(r) = j_l(\sqrt{\epsilon}r)$ where j_l is a spherical Bessel function of first kind.

(b) (2p) Using Rodrigues formula, show from the representation of spherical Bessel functions of first kind, namely

$$j_l(\rho) = \frac{\rho^l}{2^{l+1}l!}\int_{-1}^{1} e^{i\rho s}\ (1-s^2)^l\ ds \tag{13}$$

that their link with the Legendre polynomials P_l:

$$j_l(\rho) = \frac{1}{2i^l}\int_{-1}^{1} e^{i\rho s}\ P_l(s)\ ds \tag{14}$$

(c) (1p) For $U = 0$, obtain the 3D plane wave solution of (11).

(d) *Bonus:* (1 bp) Justify the dispersion relation of the 3D plane waves: $E = \frac{(\hbar\boldsymbol{k})^2}{2m}$. Moreover, a dispersion relation is normally a relation between the angular frequency ω and the wave number $\boldsymbol{k}$ then why is the previous relation named dispersion relation?

In the rescaled variables, one identifies $\epsilon = k^2$. So far, you have got two sets of orthogonal functions solutions of (11): the plane waves parametrized by $\boldsymbol{k}$ (infinitely uncountable set) and the functions $j_l(kr)\, Y_{lm}(\theta, \varphi)$ parametrized by l and m (infinitely countable set). Let us now find the relation between those two sets:

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} c_{lm}(k)\ j_l(kr)\ Y_{lm}(\theta, \varphi) \tag{15}$$

where $\boldsymbol{r} = (x, y, z)$. Basically, we are looking for coefficients $c_{lm}(k)$.

(e) (0.5p) Justify that the left-hand side of (15) does not depend on φ and that in fact:

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{l=0}^{+\infty} \widetilde{c}_l(k)\ j_l(kr)\ P_l\big(\cos(\theta)\big) \tag{16}$$

(f) (1.5p) Find the coefficients $\widetilde{c}_l(k)$ using the orthogonality of Legendre polynomials; be careful with the variables of integration. You should eventually get the the fantastic relation:

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{l=0}^{+\infty} (2l+1)\ i^l\ j_l(kr)\ P_l\big(\cos(\theta)\big) \tag{17}$$