Mathematical Methods for Physicists, FK7048, fall 2018
Exercise sheet 4, **due monday october 1st, 10:00**

This is a hard deadline since we want to have the possibility to discuss hand-in problems at the lecture. We will not accept hand-ins after this deadline. Email your nice solutions to Eddy (ardonne@fysik.su.se). Remember to write your email address on the top of the first page of your solutions! Scanned solutions will be accepted if, and only if, the quality is good enough to be read and corrected. Please submit solutions in electronic form in **one PDF file** only.
Total amount of points: 13 p + 1 bp.

Similarity solutions

When a PDE is dimensionally consistent, one can look for solutions which only depend on a dimensionless combination of the independent variables. You will do so in exercises 1 and 2.

1 The Heat equation

The heat equation in 1D is:

$$\frac{\partial u}{\partial t} = \sigma \, \frac{\partial^2 u}{\partial x^2} \qquad \sigma = cst \tag{1}$$

(a) (0.5 p) Given that $[t] = T$ and $[x] = L$, what is the dimension of σ? Infer a dimensionless variable $z \cong x \; t^{\alpha} \; \sigma^{\beta}$.

(b) (1 p) Equation (1) being homogeneous, there is no constraint on the dimension of the dependent variable. So, one can assume that $[u] = T^{\delta}$. Look for solutions of the form $u(x,t) = t^{\delta} \; U(z)$ and derive the ODE for $U(z)$.

(c) (2 p) In (b) you should have obtained a 2nd order linear ODE for $U(z)$. Solve it by the Frobenius method in the case $\delta = -\frac{1}{2}$. Conclude on the similarity solution of the heat equation. *Hint*: discard the series which has only odd powers. It is possible to impose only even powers to the other series, that you should be able to obtain in closed form.

2 Viscous flow

Here are the Prandtl's equations describing a viscous flow over a flat plate at $y = 0$ (see figure 1):

$$u(x,y) \, \frac{\partial u}{\partial x} + v(x,y) \, \frac{\partial u}{\partial y} = \nu \, \frac{\partial^2 u}{\partial y^2} \qquad \nu = cst \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

u and v are components of the velocity field in the half-plane $-\infty < x < +\infty \; , \; y \geq 0$; ν is the kinematic viscosity. These equations are valid on a certain length scale above the plate, this region is called the laminar boundary layer.

(a) (0.5 p) From equations (2) and (3), establish a 3rd order PDE for the stream function $w(x,y)$ defined by:

$$\begin{cases} u = \frac{\partial w}{\partial y} \\ v = -\frac{\partial w}{\partial x} \end{cases} \tag{4}$$

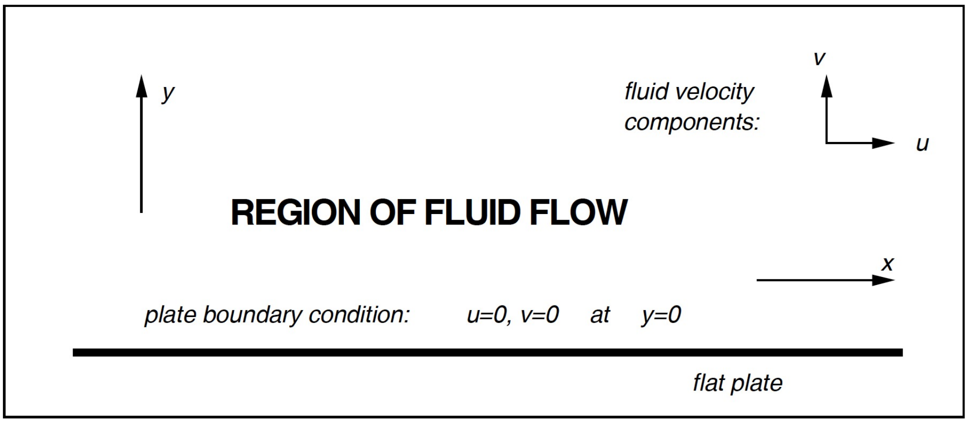

Figure 1: Flow over a flat plate at $y = 0$

Afar from the plate, the velocity of the fluid is expected to be constant. Hence, we impose the boundary condition $\lim_{y \to \infty} \frac{\partial w}{\partial y} = U$. This suggests that:

$$[w] = [U][y] \tag{5}$$

where $[X]$ means the scale of the physical quantity X.

(b) (1 p) Balancing terms in the equation for w, find a relation between $[w]$, $[x]$, $[y]$ and $[\nu]$. Then using equation (5), show that:

$$[y] = \left[\sqrt{\frac{\nu x}{U}}\right] \quad \text{and} \quad [w] = \left[\sqrt{U \nu x}\right] \tag{6}$$

Note: $[XY] = [X][Y]$

(c) (1 p) The previous result encourages to look for similarity solutions of the form $w(x, y) = \sqrt{U \nu x}\, f(\eta)$ with $\eta \cong y\sqrt{\frac{U}{\nu x}}$. Derive a 3rd order non-linear ODE for $f(\eta)$. It is know as the Blasius equation. One can only solve it numerically, which however costs less than solving directly the Prandtl's equations.

3 Method of characteristics

(2 p) Solve the following quasi-linear 1st order PDE by the method of characteristics:

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2(x, y) \\ u(x = 0, y) = 1 \quad \forall y \in \mathbb{R} \end{cases} \tag{7}$$

Hint: A nice parametrization of the initial conditial is $(x = 0; y = t; u = 1)$, $\forall t \in \mathbb{R}$.

4 Complex potential of a 2D incompressible irrotational flow

Let us consider a 2D incompressible irrotational flow with velocity field $\boldsymbol{u}$.
On the one hand, incompressibility ($\operatorname{div} \boldsymbol{u} = 0$) implies that there is a so-called stream function $\psi(x, y)$

such that $\boldsymbol{u} = \boldsymbol{curl}(\psi\ \boldsymbol{e}_z)$. On the other hand, due to irrotationality ($\boldsymbol{curl}(\boldsymbol{u}) = \boldsymbol{0}$) there also exists a potential function $\phi(x, y)$ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. In the end:

$$\begin{cases} u_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ u_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{cases} \quad \text{or} \quad \begin{cases} u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\ u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} \end{cases} \quad \text{in polar coordinates.} \tag{8}$$

One defines the complex potential on an open set of $\mathbb{C}$ as follows:

$$\begin{aligned} f\ :\ &\mathcal{U} \to \mathbb{C} \\ &z\ \mapsto\ \phi(x, y) + i\ \psi(x, y), \quad z \equiv x + i\ y \end{aligned}$$

(a) (0.5 p) Show that f is a holomorphic (i.e., analytic) function and give an expression of its derivative $f'(z)$ in terms of u_x and u_y.

(b) (0.5 p) Derive the following relation:

$$\oint f'(z)dz = \oint \boldsymbol{u} \cdot \boldsymbol{dl} + i \oint \left(\boldsymbol{u} \times \boldsymbol{dl}\right) \cdot \boldsymbol{e}_z \tag{9}$$

where the integration is over a closed contour in the complex plane, $\boldsymbol{dl} = dx\ \boldsymbol{e}_x + dy\ \boldsymbol{e}_y + dz\ \boldsymbol{e}_z$ and the dot stands for the euclidian scalar product.
The real part of (9) is the circulation of $\boldsymbol{u}$ along the contour whereas the imaginary part is the flux of $\boldsymbol{u}$ across this contour.

(c) (2 p) For the following complex potentials, use the residue theorem to compute the circulation and the flux, find the velocity field and plot its field lines:

(c_1) Vortex flow: $f(z) = -i\frac{\Gamma}{2\pi}\,\mathrm{Log}(z)$. Is the flow irrotational everywhere?

(c_2) Source/sink flow: $f(z) = \frac{\lambda}{2\pi}\,\mathrm{Log}(z)$.

Bonus question (1 bp): Apply the Green(-Riemann) formula to the imaginary part of (9). Is the source/sink flow incompressible everywhere?

5 An inhomogeneous ODE: second round

You have been asked to solve this boundary value problem in sheet 2 and the variation of parameters was the only method you knew. You are now going to find back your solutions with a powerful tool: Green's functions.

$$\begin{cases} y''(x) + 4y(x) = e^{\frac{x}{2}} \\ y(0) = 0 \quad \text{and} \quad y\left(\frac{\pi}{4}\right) = 0 \end{cases} \tag{10}$$

(a) (0.5 p) What is a self-adjoint equation? What does "homogeneous boundary conditions" mean? Why does it matter in the Green's functions method?

(b) (1.5 p) Construct the Green's function associated to (10) and use it to find the solution of this boundary value problem. Justify the steps of your construction briefly.