Mathematical Methods for Physicists, FK7048, fall 2018
Exercise sheet 2, **due monday september 17th, 10:00**

This is a hard deadline since we want to have the possibility to discuss hand-in problems at the lecture. We will not accept hand-ins after this deadline. Email your nice solutions to Eddy (ardonne@fysik.su.se). Remember to write your email address on the top of the first page of your solutions! Scanned solutions will be accepted if, and only if, the quality is good enough to be read and corrected. Please submit solutions in electronic form in **one PDF file** only.
Total amount of points: 13 p, and 1 bonus points (1 bp).

1 Reduction of order

(2p) For a linear ODE of any order, if you know one solution y_1 then you can reduce the order of the ODE by performing the substitution $y = y_1 u$. Do this for a general 2nd order ODE:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1}$$

Solve the equation that you get for u. You should eventually find back the integral expression seen in class of the second solution generated from the first.

2 A scale invariant equation

An ODE is said to be scale invariant if there exists a $p \in \mathbb{R}$ such that the transformation $(x \to ax; y \to a^p y)$ leaves the ODE unchanged. This is a generalization of isobaric equations (which are 1st order ODEs). For a scale invariant ODE, one can perform the transformation $y(x) = x^p u(x)$ to obtain an ODE for $u(x)$ which is, by construction, equidimensional in x. That is, an equation that is invariant under the transformation $x \to ax$. It is known that an equidimensional equation can be transformed into an autonomous equation for $u(t)$ thanks to the change of variable $x = e^t$. We remind that, by definition, an autonomous ODE does not depend explicitly on the independent variable (here t). This means that it is possible to reduce its order by setting $w(u) = u'(t)$. This yields a simpler ODE for $w(u)$, that does not explicitly depend on t.

(a) (4p) Apply this recipe the following scale invariant equation:

$$xy''(x) + \big(2 - xy(x)\big)y'(x) = y^2(x) \tag{2}$$

to find the differential equation for $w(u)$, namely $ww' - w = uw$.

(b) (1bp) Perform the integrals, to show that the general solution $y(x)$ can be written as

$$y(x) = \frac{c_1}{x}\tan\big[(\frac{c_1}{2}\ln x + c_2)\big] - \frac{1}{x}\ .$$

3 Equivalence between 2nd order linear ODEs and Riccati equations

Any 2nd order linear ODE is equivalent to a Riccati equation. We are going to use this fact to solve the following equation:

$$y''(x) + 2xy'(x) + (2x + 1)y(x) = 0\ . \tag{3}$$

(a) (1 p) Perform the transformation $y(x) = e^{u(x)}$ and show that $v(x) = u'(x)$ satisfies a Riccati equation.

(b) (1 p) Look for a particular solution of this Riccati equation which has the simple form $v_0(x) = ax + b, \quad (a, b) \in \mathbb{C}^2$.

(c) (1 p) Substitute $v(x) = w(x) + v_0(x)$ in the Riccati equation. You should recognize the kind of equation that you get for $w(x)$. Apply the trick seen in class to transform this last equation into a 1st order linear non homogeneous ODE.

(d) (1 p) Solve this non homogeneous ODE by the method of the integrating factor. Write your solution in terms of the function $\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt$. One can now write the solution for (3) as an integral expression, but you do not have to do this.

4 An inhomogeneous ODE

(3p) Let us consider the following 2nd order linear non homogenous ODE with its boundary conditions:

$$\begin{cases} y''(x) + 4y(x) = e^{\frac{x}{2}} \\ y(0) = 0 \quad \text{and} \quad y\left(\frac{\pi}{4}\right) = 0 \end{cases} \tag{4}$$

Solve the differenctial equation (4).