Written Examination for Mathematical Methods of Physics
2014.11.08 at 09:00-14:00

Allowed help: "Arfken and Weber" (or "Weber and Arfken"), "Physics Handbook", "Beta: Mathematics Handbook", lecture notes from the course.

In order to get full credit:
1) Used formalisms should be clearly defined
2) All steps in your derivations that are based on references in the above books should be clearly given through reference to the relevant equations or tables.

1. Use calculus of residues to evaluate the integral $\int_0^\infty \frac{\cos(\pi x)dx}{x^2+1}$. (2p)

2. Calculate the Fourier transform of $f(x) = 1 - |x/2|$ for $-2 \le x \le 2$, with $f(x) = 0$ elsewhere. (3p)

3. Use the Frobenius method to find the two independent solutions to the equation:

$$x(1-x)\frac{dy^2}{dx^2} - x\frac{dy}{dx} - y = 0$$ (4p)

4. Express the Legendre functions $P_2(x)$, $P_3(x)$ and $P_4(x)$ in terms of $P_0(x)$ and $P_1(x)$. (3p)

5. A sphere initially at 0° has its surface kept at 100° from $t=0$ on (for example a frozen pea in boiling water). Find the time-dependent temperature distribution. (Hint: Subtract 100° from all temperatures and solve the problem; then add the 100° to the answer). (4p)

6. Use Laplace transforms to solve $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = at^2$, where $y(t) = 0$ for $t < 0$ and $y(0) = 1, y'(0) = 3$. For the maximum points the back transform should use the Bromwich integral. (4p)

7. Calculate the Green's function for the differential equation $\left(\frac{d^2}{dx^2} - \lambda^2\right)y(x) = R(x)$ on the interval $(-\infty, \infty)$ where the boundary conditions are $y(-\infty) = y(\infty) = 0$. (3p)

8. Show that $J_0(x)$ on the form $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\vartheta)d\vartheta$ solves the Bessel equation for n=0. (3p)

9. Use the Laplace transform technique and the defining equation of the Green's function to find the Green's function for the equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = f(t)$, where $y(t) = 0$ for $t < 0$ and $y(0) = 1, y'(0) = 3$. Use this Green's function to solve the equation for the case $f(t) = at^2$. (4p)

1. Use calculus of residues to evaluate the integral $\int_0^\infty \frac{cos(\pi x)dx}{x^2+1}$. (2p)

1.
$$\int_0^\infty \frac{\cos(\pi x)}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos(\pi x)}{x^2+1}dx = \frac{1}{2}\mathrm{Re}\int_{-\infty}^{\infty}\frac{e^{i\pi x}}{x^2+1}dx$$

Poles at $z=\pm i$. Close in upper half plane and use Jordan's lemma. The pole at $z=i$ is contained

$$\mathrm{Res}(z=i) = \lim_{z\to i}\frac{(z-i)e^{i\pi z}}{(z-i)(z+i)} = \frac{e^{+\pi}}{2i}$$

$$\therefore \int_0^\infty \frac{\cos(\pi x)}{x^2+1}dx = \frac{1}{2}\mathrm{Re}\left\{2\pi i\cdot\frac{e^{-\pi}}{2i}\right\} = \frac{\pi}{2}e^{-\pi}$$

2. Calculate the Fourier transform of f(x) = 1 − |x/2| for −2 ≤ x ≤ 2, with f(x) = 0 elsewhere.

(3p)

$$f(x) = 1 - |x/2| = \begin{cases} 1 + \frac{x}{2} & -2 \le x \le 0 \\ 1 - \frac{x}{2} & 0 \le x \le 2 \end{cases} \qquad f(x) = 0 \text{ elsewhere}$$

Fourier transform $\tilde{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{itx} dx =$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-2}^{0} \left(1 + \frac{x}{2}\right) e^{itx} dx + \int_{0}^{2} \left(1 - \frac{x}{2}\right) e^{itx} dx \right\} =$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-2}^{2} e^{itx} dx + \frac{1}{2} \int_{-2}^{0} x e^{itx} dx - \frac{1}{2} \int_{0}^{2} x e^{itx} dx \right\} \equiv \frac{1}{\sqrt{2\pi}} \left\{ I + \frac{1}{2} II - \frac{1}{2} III \right\}$$

$$I \equiv \int_{-2}^{2} e^{itx} dx = \frac{1}{it} \left[e^{itx} \right]_{-2}^{2} = \frac{1}{it} \left(e^{2it} - e^{-2it} \right)$$

$$II \equiv \int_{-2}^{0} x e^{itx} dx = \frac{1}{it} \left[x e^{itx} \right]_{-2}^{0} - \frac{1}{it} \int_{-2}^{0} e^{itx} dx = \frac{2}{it} e^{-2it} + \frac{1}{t^2} \left[e^{itx} \right]_{-2}^{0} =$$

$$= \frac{2}{it} e^{-2it} + \frac{1}{t^2} \left(1 - e^{-2it} \right)$$

$$III \equiv \int_{0}^{2} x e^{itx} dx = \frac{1}{it} \left[x e^{itx} \right]_{0}^{2} - \frac{1}{it} \int_{0}^{2} e^{itx} dx = \frac{2}{it} e^{2it} + \frac{1}{t^2} \left(e^{2it} - 1 \right)$$

$$I + \frac{1}{2} II - \frac{1}{2} III = -\frac{1}{2t^2} \left(e^{2it} - 2 + e^{-2it} \right) = -\frac{1}{2t^2} \left(e^{it} - e^{-it} \right)^2 =$$

$$= -\frac{1}{2t^2} (2i)^2 \left(\frac{e^{it} - e^{-it}}{2i} \right)^2 = \frac{2}{t^2} \sin^2 t$$

$$\therefore \tilde{f}(t) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin t}{t} \right)^2 = \frac{1}{\sqrt{2\pi}} \frac{1 - \cos 2t}{t^2}$$

3. Use the Frobenius method to find the two independent solutions to the equation:

$$x(1-x)\frac{dy^2}{dx^2} - x\frac{dy}{dx} - y = 0 \qquad \text{(4p)}$$

* The problem was corrected to require only <u>one</u> solution *

The equation $x(1-x)y'' - xy' - y = 0$ has regular singularities at $x=0$ and $x=1$. Most easily seen from writing

$$y'' - \frac{1}{1-x}y' - \frac{1}{x(1-x)}y = 0$$

Make the ansatz $y(x) = \sum_{k=0}^{\infty} a_k x^{k+s}$, $y' = \sum_{k=0}^{\infty} a_k (k+s) x^{k+s-1}$ and

$y'' = \sum_{k=0}^{\infty} a_k (k+s)(k+s-1) x^{k+s-2}$ which gives the equation

$$0 = x(1-x)y'' - xy' - y = x(1-x)\sum_{k=0}^{\infty} a_k(k+s)(k+s-1)x^{k+s-2} - x\sum_{k=0}^{\infty} a_k(k+s)x^{k+s-1}$$

$$- \sum_{k=0}^{\infty} a_k x^{k+s} = \sum_{k=0}^{\infty} a_k (k+s)(k+s-1)x^{k+s-1} - \sum_{k=0}^{\infty} a_k(k+s)(k+s-1)x^{k+s}$$

$$- \sum_{k=0}^{\infty} a_k (k+s)x^{k+s} - \sum_{k=0}^{\infty} a_k x^{k+s} = a_0 s(s-1)x^{s-1} +$$

$$+ \sum_{k=0}^{\infty} x^{k+s}\left\{ a_{k+1}(k+s)(k+s+1) - a_k \underbrace{\left[(k+s)(k+s-1) + k+s+1\right]}_{(k+s)^2+1} \right\}$$

Indicial equation $s(s-1) = 0 \Rightarrow s=0$ or $s=1$

According to Fuch's theorem only $s=1$ can be guaranteed to give a solution.

$s=1$: $$a_{k+1} = \frac{(k+1)^2+1}{(k+1)(k+2)} a_k$$

$$a_1 = \frac{1+1}{1\cdot 2} a_0 = a_0$$

$$a_2 = \frac{2^2+1}{2\cdot 3} a_1 = \frac{5}{3\cdot 2} a_0$$

$$a_3 = \frac{3^2+1}{3\cdot 4} a_2 = \frac{10\cdot 5}{4\cdot 3\cdot 3\cdot 2} a_0$$

This gives $y(x) = a_0 x\left\{1 + \sum_{n=1}^{\infty} x^n \prod_{k=0}^{n-1} \frac{(k+1)^2+1}{(k+1)(k+2)}\right\}$

For $s=0$ the recurrence relation becomes

$a_{k+1} = \frac{k^2+1}{k(k+1)} a_k$ which diverges for $k=0$

x ———————————————————————— x

Check that $y(x)$ for $s=1$ is a solution. Go up to x^5

The factors in the product are

$k=$	0	1	2	3
	$\frac{1^2+1}{1\cdot 2} = 1$	$\frac{2^2+1}{2\cdot 3} = \frac{5}{6}$	$\frac{3^2+1}{3\cdot 4} = \frac{10}{3\cdot 4} = \frac{5}{6}$	$\frac{4^2+1}{4\cdot 5} = \frac{17}{4\cdot 5}$

$a_1 = a_0,\ a_2 = a_0 \cdot \frac{5}{6} = \frac{5}{6}a_0,\ a_3 = \frac{5}{6}a_2 = \frac{25}{36}a_0,\ a_4 = \frac{17}{4\cdot 5}a_3 = \frac{25\cdot 17}{4\cdot 5\cdot 36} = \frac{5\cdot 17}{4\cdot 36}$

$$y(x) = a_0\left\{x + x^2 + \frac{5}{6}x^3 + \frac{25}{36}x^4 + \frac{5\cdot 17}{4\cdot 6\cdot 6}x^5\right\}$$

$$y'(x) = a_0\left\{1 + 2x + \frac{5}{2}x^2 + \frac{25}{9}x^3 + \frac{25\cdot 17}{4\cdot 6\cdot 6}x^4\right\}$$

$$y''(x) = a_0\left\{2 + 5x + \frac{25}{3}x^2 + \frac{25\cdot 17}{6\cdot 6}x^3\right\}$$

$$xy'' - x^2y'' - xy' - y = a_0\left\{2x + 5x^2 + \frac{25}{3}x^3 + \frac{25\cdot 17}{6\cdot 6}x^4 - 2x^2 - 5x^3 - \frac{25}{3}x^4 - \frac{25\cdot 17}{6\cdot 6}x^5 - x - 2x^2 - \frac{5}{2}x^3 - \frac{25}{9}x^4 - \frac{25\cdot 17}{4\cdot 6\cdot 6}x^5 - x - x^2 - \frac{5}{6}x^3 - \frac{25}{36}x^4 - \frac{5\cdot 17}{4\cdot 6\cdot 6}x^5\right\}$$

$$= a_0\left\{x\underbrace{(2-1-1)}_{=0} + x^2\underbrace{(5-2-2-1)}_{=0} + x^3\underbrace{\left(\frac{25}{3} - 5 - \frac{5}{2} - \frac{5}{6}\right)}_{=0} + x^4\underbrace{\left(\frac{25\cdot 17}{6\cdot 6} - \frac{25}{3} - \frac{25}{9} - \frac{25}{36}\right)}_{=0} + O(x^5)\right\} = O(x^5) \text{ ok}$$

$$\frac{25}{3} - \frac{5}{2} - 5 - \frac{5}{6} = \frac{50 - 15 - 30 - 5}{6} = 0$$

$$\frac{25\cdot 17}{6\cdot 6} - \frac{25}{3} - \frac{25}{9} - \frac{25}{36} = \frac{25\cdot 17 - 25\cdot 12 - 25\cdot 4 - 25\cdot 1}{36} = 0$$

4. Express the Legendre functions $P_2(x)$, $P_3(x)$ and $P_4(x)$ in terms of $P_0(x)$ and $P_1(x)$. (3p)

Use the recurrence relation $P_{n+1} = \frac{2n+1}{n+1} x P_n - \frac{n}{n+1} P_{n-1}$

starting from $P_0(x) = 1$ and $P_1(x) = x$

$$P_2(x) = \frac{2 \cdot 1 + 1}{1+1} x P_1 - \frac{1}{1+1} P_0 = \frac{3}{2} x P_1 - \frac{1}{2} P_0$$

$$P_3(x) = \frac{2 \cdot 2 + 1}{2+1} x P_2 - \frac{2}{3} P_1 = \frac{5}{3} x P_2 - \frac{2}{3} P_1 =$$

$$= \frac{5}{3} x \left\{ \frac{3}{2} x P_1 - \frac{1}{2} P_0 \right\} - \frac{2}{3} P_1 = \frac{5}{2} x^2 P_1 - \frac{5}{6} x P_0 - \frac{2}{3} P_1 =$$

$$= \left(\frac{5}{2} x^2 - \frac{2}{3} \right) P_1 - \frac{5}{6} x P_0 = \left\{ \frac{5}{2} x^2 - \frac{4}{6} - \frac{5}{6} \right\} P_1 = \left(\frac{5}{2} x^2 - \frac{3}{2} \right) P_1$$

$$P_4(x) = \frac{2 \cdot 3 + 1}{4} x P_3 - \frac{3}{4} P_2 = \frac{7}{4} x \left(\frac{5}{2} x^2 - \frac{3}{2} \right) P_1 - \frac{3}{4} \left(\frac{3}{2} x P_1 - \frac{1}{2} P_0 \right) =$$

$$= \left\{ \frac{35}{8} x^3 - \frac{21}{8} x \right\} P_1 - \frac{9}{8} x P_1 + \frac{3}{8} P_0 = \frac{1}{8} \left\{ 35 x^3 - 30 x \right\} P_1 + \frac{3}{8} P_0$$

The way the problem was formulated also powers of $P_1(x)$ had to be accepted, i.e.

$$P_2(x) = \frac{3}{2} \left[P_1(x) \right]^2 - \frac{1}{2} P_0(x)$$

$$P_3(x) = \frac{5}{2} \left[P_1(x) \right]^3 - \frac{3}{2} P_1(x)$$

$$P_4(x) = \frac{35}{8} \left[P_1(x) \right]^4 - \frac{30}{8} \left[P_1(x) \right]^2 + \frac{3}{8} P_0(x)$$

5. A sphere initially at 0° has its surface kept at 100° from $t=0$ on (for example a frozen pea in boiling water). Find the time-dependent temperature distribution. (Hint: Subtract 100° from all temperatures and solve the problem; then add the 100° to the answer). (4p)

The heat equation in spherical polar coordinates

$\frac{\partial u}{\partial t} = a^2 \nabla^2 u$. The symmetric boundary conditions lead to dependence only on r and t

Thus, $\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} \right)$

Make the boundary condition homogeneous through

$v(r,t) = u(r,t) - 100$ and solve for $v(r,t)$

Separate variables $v(r,t) = T(t) \cdot R(r)$

$$\frac{1}{a^2 T}\frac{dT}{dt} = \frac{1}{R}\frac{d^2R}{dr^2} + \frac{2}{rR}\frac{dR}{dr} = -\lambda^2$$

The equation for T: $\frac{dT}{dt} + \lambda^2 a^2 T = 0 \rightarrow T(t) = Ce^{-\lambda^2 a^2 t}$

The equation for r: $\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \lambda^2 R = 0$

scale by r^2: $r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \lambda^2 r^2 R = 0$ spherical Bessel

$\Rightarrow R(r) = j_0(\lambda r)$ with $y_0(\lambda r)$ excluded since $r=0$ is included.

The B.C. $R(r_0) = 0 = j_0(\lambda r_0) \Rightarrow \lambda r_0 = \alpha_{0m}$

with α_{0m} a zero for j_0 and $\lambda = \frac{\alpha_{0m}}{r_0} = \frac{m\pi}{r_0}$ since $j_0(x) = \frac{\sin x}{x}$

The general solution for $v(r,t)$:

$$v(r,t) = \sum_{m=1}^{\infty} A_m\, j_0\!\left(m\pi \frac{r}{r_0}\right) e^{-\left(\frac{m\pi a}{r_0}\right)^2 t} =$$

$$= \sum_{m=1}^{\infty} \frac{r_0 A_m}{r\, m\pi} \sin\!\left(m\pi\frac{r}{r_0}\right) e^{-\left(\frac{m\pi a}{r_0}\right)^2 t}$$

$$v(r,0) = -100 = \sum_{m=1}^{\infty} A_m j_0\left(m\pi \frac{r}{r_0}\right)$$

Use the orthogonality $\displaystyle\int_0^{r_0} j_0\left(m\pi\frac{r}{r_0}\right) j_0\left(n\pi\frac{r}{r_0}\right) r^2 dr = \frac{r_0^3}{2}\left[j_1(m\pi)\right]^2 \delta_{nm}$

and project: $\displaystyle A_k \frac{r_0^3}{2}\left[j_1(k\pi)\right]^2 = -100 \int_0^{r_0} j_0\left(k\pi\frac{r}{r_0}\right) r^2 dr$

Use the recurrence relation $f_{n-1} x^{n+1} = \frac{d}{dx}\left[x^{n+1} f_n\right]$:

$$\int_0^{r_0} j_0\left(k\pi\frac{r}{r_0}\right) r^2 dr = \begin{bmatrix} u = \frac{k\pi r}{r_0} \\ r = \frac{r_0}{k\pi} u \\ dr = \frac{r_0}{k\pi} du \\ r = r_0 \to u = k\pi \end{bmatrix} = \left(\frac{r_0}{k\pi}\right)^3 \int_0^{k\pi} j_0(u) u^2 du =$$

$$= \left(\frac{r_0}{k\pi}\right)^3 \int_0^{k\pi} \frac{d}{du}\left[u^2 j_1(u)\right] du = \frac{r_0^3}{k\pi} j_1(k\pi)$$

$$\therefore\quad A_k = -\frac{100 \cdot 2}{r_0^3\left[j_1(k\pi)\right]^2} \cdot \frac{r_0^3}{k\pi} j_1(k\pi) =$$

$$= -\frac{200}{\pi} \cdot \frac{1}{k j_1(k\pi)}$$

$$\therefore\quad u(r,t) = 100 - \frac{200}{\pi} \sum_{k=1}^{\infty} \frac{j_0\left(k\pi\frac{r}{r_0}\right)}{k j_1(k\pi)} e^{-\left(\frac{k\pi a}{r_0}\right)^2 t}$$

Use $j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$ to find $j_1(k\pi) = -\frac{\cos k\pi}{k\pi} = \frac{(-1)^{k+1}}{k\pi}$

and $\displaystyle u(r,t) = 100 - 200 \sum_{k=1}^{\infty} (-1)^{k+1} j_0\left(k\pi\frac{r}{r_0}\right) e^{-\left(\frac{k\pi a}{r_0}\right)^2 t} =$

$$= 100 + \frac{200 r_0}{\pi} \sum_{k=1}^{\infty} (-1)^k \frac{\sin\left(\frac{k\pi r}{r_0}\right)}{k r} e^{-\left(\frac{k\pi a}{r_0}\right)^2 t}$$

6. Use Laplace transforms to solve $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = at^2$, where $y(t) = 0$ for $t < 0$ and $y(0) = 1, y'(0) = 3$. For the maximum points the back transform should use the Bromwich integral.

(4p)

$y'' - 3y' + 2y = at^2$, $y(t) = 0$ for $t < 0$ and $y(0) = 1, y'(0) = 3$

The Laplace transform of the derivative gives

$$\int_0^\infty e^{-st} y' dt = \left[y e^{-st} \right]_0^\infty + s \int_0^\infty e^{-st} y \, dt = s\tilde{y} - y(0)$$

$$\int_0^\infty e^{-st} y'' dt = \left[y' e^{-st} \right]_0^\infty + s \int_0^\infty e^{-st} y' dt = s^2 \tilde{y} - s y(0) - y'(0)$$

Right hand side: $a \int_0^\infty e^{-st} t^2 dt = a \left[-\frac{1}{s} e^{-st} t^2 \right]_0^\infty + \frac{2a}{s} \int_0^\infty e^{-st} t \, dt =$

$$= \frac{2a}{s} \left[-\frac{1}{s} e^{-st} t \right]_0^\infty + \frac{2a}{s^2} \int_0^\infty e^{-st} dt = \frac{2a}{s^2} \left[-\frac{1}{s} e^{-st} \right]_0^\infty = \frac{2a}{s^3}$$

Transformed equation:

$$s^2 \tilde{y} - \underbrace{s y(0)}_{-s} - \underbrace{y'(0)}_{-3} - 3s\tilde{y} + \underbrace{3y(0)}_{+3} + 2\tilde{y} = \frac{2a}{s^3}$$

$\left. \begin{array}{l} y(0) = 1 \\ y'(0) = 3 \end{array} \right\}$ $(s^2 - 3s + 2)\tilde{y} = \frac{2a}{s^3} + s$

$$\tilde{y} = \frac{2a}{s^3(s^2 - 3s + 2)} + \frac{s}{s^2 - 3s + 2}$$

Look for poles $s^2 - 3s + 2 = 0 \Rightarrow (s - \frac{3}{2})^2 = \frac{1}{4} \Rightarrow s = \begin{cases} 2 \\ 1 \end{cases}$

for $\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{s e^{st}}{(s-2)(s-1)} ds$ close in the left half plane and choose $\gamma > 2$

The poles at $s = 2$ and $s = 1$ contribute

$$\text{Res}(s=2) = \left. \frac{s e^{st}}{s-1} \right|_{s=2} = 2e^{2t}$$

$$\text{Res}(s=1) = \left. \frac{s e^{st}}{s-2} \right|_{s=1} = -e^t$$

For the second integral we use the same contour and find

$$\operatorname{Res}(s=2) = \left.\frac{e^{st}}{s^3(s-1)}\right|_{s=2} = \frac{e^{2t}}{8}$$

$$\operatorname{Res}(s=1) = \left.\frac{e^{st}}{s^3(s-2)}\right|_{s=1} = -e^{t}$$

We have a triple pole at $s=0$

$$\operatorname{Res}(s=0) = \frac{1}{2}\frac{d^2}{ds^2}\left.\left(\frac{s^3 e^{st}}{s^3(s-2)(s-1)}\right)\right|_{s=0} \qquad (*) \text{ see alternative}$$

$$\frac{d}{ds}\left(\frac{e^{st}}{(s-2)(s-1)}\right) = \frac{e^{st}}{(s-2)(s-1)}\left\{t - \frac{1}{s-2} - \frac{1}{s-1}\right\}$$

$$\frac{d^2}{ds^2}\left(\frac{e^{st}}{(s-2)(s-1)}\right) = \frac{e^{st}}{(s-2)(s-1)}\left\{t - \frac{1}{s-2} - \frac{1}{s-1}\right\}^2 + \frac{e^{st}}{(s-2)(s-1)}\left\{\frac{1}{(s-2)^2} + \frac{1}{(s-1)^2}\right\}$$

$$\frac{1}{2}\cdot\frac{d^2}{ds^2}\left.\left(\frac{e^{st}}{(s-2)(s-1)}\right)\right|_{s=0} = \frac{1}{2}\left\{\frac{1}{2}\left(t+\frac{1}{2}+1\right)^2 + \frac{1}{2}\left(\frac{1}{4}+1\right)\right\} =$$

$$= \frac{1}{4}\left\{\left(t+\frac{3}{2}\right)^2 + \frac{5}{4}\right\} = \frac{1}{4}\left\{t^2+3t+\frac{14}{4}\right\} = \frac{t^2}{4} + \frac{3}{4}t + \frac{14}{16}$$

and $$y(t) = 2a\left\{\frac{t^2}{4} + \frac{3}{4}t + \frac{14}{16} + \frac{e^{2t}}{8} - e^{t}\right\} + 2e^{2t} - e^{t}$$

$$y(0) = 2a\left\{\frac{14}{16} + \frac{1}{8} - 1\right\} + 2 - 1 = 1$$

$$y'(0) = 2a\left\{\frac{t}{2} + \frac{3}{4} + \frac{e^{2t}}{4} - e^{t}\right\} + 4e^{2t} - e^{t}\Big|_{t=0} = 2a\left(\frac{3}{4}+\frac{1}{4}-1\right) + 3 = 3$$

Check the solution

$$y'' = 2a\left(\frac{1}{2} + \frac{e^{2t}}{2} - e^{t}\right) + 8e^{2t} - e^{t}$$

$$y'' - 3y' + 2y = a + ae^{2t} - 2ae^{t} + 8e^{2t} - e^{t} - 6a\left(\frac{t}{2} + \frac{3}{4} + \frac{e^{2t}}{4} - e^{t}\right) - 12e^{2t} + 3e^{t} +$$

$$+ 4a\left\{\frac{t^2}{4} + \frac{3}{4}t + \frac{14}{16} + \frac{e^{2t}}{8} - e^{t}\right\} + 4e^{2t} - 2e^{t} =$$

6.

$$= a + ae^{2t} - 2ae^{t} - \cancel{3at} - \frac{9}{2}a - \frac{3a}{2}e^{2t} + 6ae^{t} - 12e^{2t} + 3e^{t} +$$

$$+ at^2 + \cancel{3at} + \frac{14a}{4} + \frac{ae^{2t}}{2} - 4ae^{t} + 4e^{2t} - 2e^{t} + 8e^{2t} - e^{t} =$$

$$= at^2 + \underbrace{\frac{18}{4}a - \frac{18}{4}a}_{=0} + e^{2t}\Big(a - \frac{3}{2}a \underbrace{- 12 + 12}_{=0} + \frac{a}{2}\Big) +$$

$$+ e^{t}\underbrace{(-2a + 6a + 3 - 4a - 2 - 1)}_{=0} = at^2 \qquad \text{OK}$$

(*) Laurent expansion of $\dfrac{2ae^{zt}}{z^3(z-2)(z-1)}$ around $z=0$ to find $\dfrac{a_{-1}}{z}$; Expand to z^2:

$$e^{zt} \approx 1 + zt + \frac{1}{2}(zt)^2$$

$$\frac{1}{z-2} = -\frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} \approx -\frac{1}{2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2\right)$$

$$\frac{1}{z-1} = -\frac{1}{1-z} \approx -1\left(1 + z + z^2\right)$$

Multiply together, keep terms up to z^2:

$$\frac{2ae^{zt}}{z^3(z-2)(z-1)} \approx 2a\cdot\frac{1}{2}\left(1+zt+\frac{1}{2}z^2t^2\right)\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2\right)\left(1+z+z^2\right) \approx$$

$$\approx a\left(1+zt+\frac{1}{2}z^2t^2\right)\left(1+z+z^2+\frac{z}{2}+\frac{z^2}{2}+\frac{z^2}{4}\right) = a\left(1+zt+\frac{1}{2}z^2t^2\right)\left(1+\frac{3}{2}z+\frac{7}{4}z^2\right) \approx$$

$$\approx a\left(1 + zt + \frac{1}{2}z^2t^2 + \frac{3}{2}z + \frac{3}{2}z^2t + \frac{7}{4}z^2\right)$$

The z^2 coefficient: $a_{-1} = \frac{7}{4}a + \frac{3}{2}at + \frac{1}{2}at^2$

7. Calculate the Green's function for the differential equation $\left(\frac{d^2}{dx^2} - \lambda^2\right) y(x) = R(x)$ on the interval $(-\infty, \infty)$ where the boundary conditions are $y(-\infty) = y(\infty) = 0$. (3p)

The boundary conditions $y(\pm\infty) = 0$ make a Fourier transform suitable.

The equation for the Green's function becomes

$$G'' - \lambda^2 G = \delta(x-x') \longrightarrow -(\omega^2+\lambda^2)\tilde{G} = \frac{1}{\sqrt{2\pi}} e^{i\omega x'}$$

$$\tilde{G} = -\frac{1}{\sqrt{2\pi}} \frac{e^{i\omega x'}}{\omega^2+\lambda^2}$$

The inverse

$$G(x,x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \tilde{G}(\omega, x')\, d\omega =$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(x'-x)}}{\omega^2+\lambda^2}\, d\omega \qquad \text{poles at } \omega = \pm i\lambda$$

For $x < x'$ we close in the UHP and enclose $\omega = i\lambda$

$$G_<(x,x') = -\frac{1}{2\pi} \oint_{UHP} \frac{e^{iz(x'-x)}}{z^2+\lambda^2}\, dz = -\frac{1}{2\pi} \cdot 2\pi i \,\mathrm{Res}\{z = i\lambda\}$$

$$\mathrm{Res}\{z=i\lambda\} = \left.\frac{(z-i\lambda)\, e^{iz(x'-x)}}{(z-i\lambda)(z+i\lambda)}\right|_{z=i\lambda} = \frac{e^{-\lambda(x'-x)}}{2i\lambda}$$

$$\therefore\; G_<(x,x') = -\frac{1}{\cancel{2\pi}} \cdot \cancel{2\pi i} \frac{e^{-\lambda(x'-x)}}{2\cancel{i}\lambda} = -\frac{1}{2\lambda} e^{-\lambda(x'-x)}$$

For $x > x'$ we close in the LHP and enclose $\omega = -i\lambda$

$$\mathrm{Res}\{z=-i\lambda\} = \left.\frac{\cancel{(z+i\lambda)}\, e^{iz(x'-x)}}{\cancel{(z+i\lambda)}(z-i\lambda)}\right|_{z=-i\lambda} = \frac{e^{\lambda(x'-x)}}{-2i\lambda}$$

$$\therefore\; G_>(x,x') = -\frac{1}{2\pi} \cdot (-2\pi i)\left(-\frac{1}{2i\lambda} e^{\lambda(x'-x)}\right) = -\frac{1}{2\lambda} e^{-\lambda(x-x')}$$

$$\Rightarrow\; G(x,x') = \begin{cases} -\frac{1}{2\lambda} e^{-\lambda(x'-x)}, & x < x' \\ -\frac{1}{2\lambda} e^{-\lambda(x-x')}, & x' < x \end{cases}$$

8. Show that $J_0(x)$ on the form $J_0(x) = \frac{1}{\pi}\int_0^{\pi} cos(xsin\vartheta)d\vartheta$ solves the Bessel equation for n=0.

(3p)

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\vartheta)\,d\vartheta$$

Bessel equation $(n=0)$: $x^2y'' + xy' + x^2y = 0$

$$\frac{d}{dx}\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\vartheta)\,d\vartheta = -\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\vartheta)\sin\vartheta\,d\vartheta$$

$$\frac{d^2}{dx^2}\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\vartheta)\,d\vartheta = -\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\vartheta)(\sin\vartheta)^2\,d\vartheta$$

Bessel eq. (scale by π):

$$-x^2\int_0^{\pi}\cos(x\sin\vartheta)(\sin\vartheta)^2\,d\vartheta - x\int_0^{\pi}\sin(x\sin\vartheta)\sin\vartheta\,d\vartheta +$$

$$+x^2\int_0^{\pi}\cos(x\sin\vartheta)\,d\vartheta;$$

Integrate the second integral by parts

$$\int_0^{\pi}\sin(x\sin\vartheta)\sin\vartheta\,d\vartheta = \Big[-\sin(x\sin\vartheta)\cos\vartheta\Big]_0^{\pi} +$$

$$+\int_0^{\pi}\cos(x\sin\vartheta)(x\cos\vartheta)\cos\vartheta\,d\vartheta = x\int_0^{\pi}\cos(x\sin\vartheta)(\cos\vartheta)^2 d\vartheta$$

$$\therefore\quad x^2\int_0^{\pi}\cos(x\sin\vartheta)\{\underbrace{-\sin^2\vartheta-\cos^2\vartheta+1}_{=0}\}\,d\vartheta = 0$$

9. Use the Laplace transform technique and the defining equation of the Green's function to find the Green's function for the equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = f(t)$, where $y(t) = 0$ for $t < 0$ and $y(0) = 1, y'(0) = 3$. Use this Green's function to solve the equation for the case $f(t) = at^2$.

(4p)

We need homogeneous initial conditions. Solve first for the particular solution to eliminate the inhomogeneous initial conditions.

$y'' - 3y' + 2y = 0$. Equation with constant coefficient

So $y = e^{\lambda t} \Rightarrow (\lambda^2 - 3\lambda + 2)e^{\lambda t} = 0 \Rightarrow \lambda = 2$ or 1

and $y(t) = Ae^{2t} + Be^{t}$

$y(0) = 1 = A + B \quad \Rightarrow B = 1 - A$

$y'(0) = 3 = 2A + B \Rightarrow \quad 2A + 1 - A = 3 \quad \Rightarrow A = 2, B = -1$

For initial condition $y(0) = 1$, $y'(0) = 3$ we transform according to $y(t) = u(t) + 2e^{2t} - e^{t}$ which gives $u(0) = u'(0) = 0$.

Do the Laplace transform on

$G'' - 3G' + 2G = \delta(t - t')$ with $G(0) = G'(0) = 0$

The derivatives (from problem 6):

$\mathcal{L}\{G''\} = s^2\tilde{G}$; $\mathcal{L}\{G'\} = s\tilde{G}$

$\mathcal{L}\{\delta(t-t')\} = \int_0^\infty e^{-st}\delta(t-t')\,dt = e^{-st'}$

$\Rightarrow (s^2 - 3s + 2)\tilde{G} = e^{-st'} \Rightarrow \tilde{G} = \frac{e^{-st'}}{(s-2)(s-1)}$

$\mathcal{L}^{-1}\{\tilde{G}\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{s(t-t')}}{(s-2)(s-1)}\,ds$

For $t < t'$ we close in the right half-plane and since $\gamma > 2$ no poles are enclosed and $G(t,t') = 0 \quad t < t'$

For $t > t'$ we close in the left half-plane and obtain

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{s(t-t')}}{(s-2)(s-1)}\,ds = \mathrm{Res}\{s=2\} + \mathrm{Res}\{s=1\}$$

$$\mathrm{Res}\{s=2\} = \left.\frac{e^{s(t-t')}}{s-1}\right|_{s=2} = e^{2(t-t')}$$

$$\mathrm{Res}\{s=1\} = \left.\frac{e^{s(t-t')}}{s-2}\right|_{s=1} = -e^{t-t'}$$

$$\therefore \quad G(t,t') = \begin{cases} 0, & t < t' \\ e^{2(t-t')} - e^{t-t'}, & t > t' \end{cases}$$

For the RHS at^2 we find

$$u(t) = \int_0^t G(t,t')\,at'^2\,dt' = a\int_0^t t'^2 e^{2(t-t')}\,dt' - a\int_0^t t'^2 e^{t-t'}\,dt'$$

$$= ae^{2t}\int_0^t t'^2 e^{-2t'}\,dt' - ae^{t}\int_0^t t'^2 e^{-t'}\,dt'$$

$$\int_0^t t'^2 e^{-2t'}\,dt' = \left[-\frac{1}{2}t'^2 e^{-2t'}\right]_0^t + \int_0^t t' e^{-2t'}\,dt' = -\frac{1}{2}t^2 e^{-2t} + \left[-\frac{1}{2}t' e^{-2t'}\right]$$

$$+\frac{1}{2}\int_0^t e^{-2t'}\,dt' = -\frac{1}{2}t^2 e^{-2t} - \frac{1}{2}te^{-2t} + \frac{1}{2}\left[-\frac{1}{2}e^{-2t'}\right]_0^t =$$

$$= -\frac{1}{2}t^2 e^{-2t} - \frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}$$

$$\int_0^t t'^2 e^{-t'}\,dt' = \left[-t'^2 e^{-t'}\right]_0^t + 2\int_0^t t' e^{-t'}\,dt' = -t^2 e^{-t} + 2\left[-t' e^{-t'}\right]_0^t +$$

$$+2\int_0^t e^{-t'}\,dt' = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2$$

Put together:

$$u(t) = ae^{2t}\int_0^t t'^2 e^{-2t'}dt' - ae^{t}\int_0^t t'^2 e^{-t'}dt' =$$

$$= -\frac{1}{2}at^2 - \frac{1}{2}at - \frac{a}{4} + \frac{a}{4}e^{2t} + at^2 + 2at + 2a - 2ae^{t} =$$

$$= \frac{1}{2}at^2 + \left\{-\frac{a}{2} + 2a\right\}t + \frac{7}{4}a + \frac{a}{4}e^{2t} - 2ae^{t} =$$

$$= \frac{1}{2}at^2 + \frac{3}{2}at + \frac{7}{4}a + \frac{a}{4}e^{2t} - 2ae^{t}$$

Check: $u(0) = \frac{7}{4}a + \frac{a}{4} - 2a = \frac{8}{4}a - \frac{8}{4}a = 0$

$$u'(0) = \left\{at + \frac{3}{2}a + \frac{a}{2}e^{2t} - 2ae^{t}\right\}_{t=0} = \frac{3}{2}a + \frac{a}{2} - 2a = 0$$

Same solution as in problem 6 with

$$y(t) = u(t) + 2e^{2t} - e^{t}$$