Written Examination for Mathematical Methods of Physics
2010.10.30 at 09:00-14:00

Lösningar / Solutions

1. $$\int_0^{2\pi} \frac{d\vartheta}{5+4\cos\vartheta} = \left\{ \begin{array}{l} z = e^{i\vartheta} \\ dz = ie^{i\vartheta}d\vartheta \\ \Rightarrow d\vartheta = -i\frac{dz}{z} \end{array} \quad \cos\vartheta = \frac{z+z^{-1}}{2} \right\} =$$

$$= -i\oint_{|z|=1} \frac{dz}{z\left\{5+4\left(\frac{z+z^{-1}}{2}\right)\right\}} = -i\oint_{|z|=1} \frac{dz}{5z+2z^2+2} = -\frac{i}{2}\oint_{|z|=1} \frac{dz}{z^2+\frac{5}{2}z+1}$$

Poles at $z^2+\frac{5}{2}z+1=0 \Rightarrow z_+ = -1/2$

$z_- = -2$

Only $z=z_+$ contained in unit circle with residue

$$(z+1/2)\cdot\frac{1}{(z+\frac{1}{2})(z+2)}\Bigg|_{z=-1/2} = \frac{1}{-\frac{1}{2}+2} = \frac{2}{3}$$

$$\therefore \int_0^{2\pi}\frac{d\vartheta}{5+4\cos\vartheta} = -\frac{i}{2}\oint\frac{dz}{z^2+\frac{5}{2}z+1} = -\frac{i}{2}\cdot 2\pi i\left\{\operatorname{Res}_{z=-\frac{1}{2}}\right\} =$$

$$= \frac{2\pi}{3}.$$

2. $g(x) = f(x-a)$

Show that their fourier transforms are related as : $\hat{g}(\omega) = e^{-ia\omega}\hat{f}(\omega)$

This requires the definition

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{-i\omega x}dx =$$

$$= \begin{bmatrix} u = x-a \\ du = dx \\ x = u+a \end{bmatrix} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega(u+a)}du =$$

$$= e^{-i\omega a}\hat{f}(\omega)$$

With the definition in A&W the sign is reversed and the relation becomes

$$\hat{g}(\omega) = e^{ia\omega}\hat{f}(\omega)$$

3. The recurrence relation $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$ gives directly for $J_2(x)$:

a) $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$ $(n = 1)$

b) $J_3(x) = \frac{2 \cdot 2}{x} J_2(x) - J_1(x) =$ $(n = 2)$

$$= \frac{4}{x}\left\{ \frac{2}{x} J_1(x) - J_0(x) \right\} - J_1(x) =$$

$$= J_1(x)\left\{ \frac{8}{x^2} - 1 \right\} - \frac{4}{x} J_0(x)$$

c) $J_4(x) = \frac{2 \cdot 3}{x} J_3(x) - J_2(x) =$ $(n = 3)$

$$= \frac{6}{x}\left\{ J_1(x)\left[\frac{8}{x^2} - 1\right] - \frac{4}{x} J_0(x) \right\} -$$

$$- \frac{2}{x} J_1(x) + J_0(x) =$$

$$= J_1(x)\left\{ \frac{48}{x^3} - \frac{8}{x} \right\} + J_0(x)\left\{ 1 - \frac{24}{x^2} \right\}$$

4. $2xy'' + (2-x)y' + \lambda y = 0$, $0 < x < \infty$, $y \equiv y(x)$, λ is rea

a) $y(x) = x^k \sum_{n=0}^{\infty} a_n x^n$, $a_0 = 1$

$$y'(x) = \sum_{n=0}^{\infty} a_n (k+n) x^{k+n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (k+n)(k+n-1) x^{k+n-2}$$

Insert in the equation:

$$\sum_{n=0}^{\infty} 2a_n (k+n)(k+n-1) x^{k+n-1} + 2\sum_{n=0}^{\infty} a_n (k+n) x^{k+n-1} - \sum_{m=0}^{\infty} a_m (k+m) x^{k+m} +$$

$$+ \lambda \sum_{p=0}^{\infty} a_p x^{p+k} = 0$$

Indicial equation (lowest power of x):

$n=0$: $\{2a_0 (k-1)k + 2a_0 k\} x^{k-1} = 0$

$\Rightarrow 2a_0 x^{k-1} \{k^2 - k + k\} = 0 \Rightarrow k = 0$

Thus $\sum_{l=0}^{\infty} x^l \{2 a_{l+1} l(l+1) + 2a_{l+1}(l+1) - l a_l + \lambda a_l\} = 0$

$$\Rightarrow \sum_{l=0}^{\infty} x^l \{2a_{l+1} [(l+1)^2] + (\lambda - l) a_l\} = 0$$

Recursion relation: $a_{l+1} = \dfrac{l-\lambda}{2(l+1)^2} a_l$

b) $\lambda = 2$: $a_{l+1} = \dfrac{l-2}{2(l+1)^2} a_l$

$a_0 = 1$

$a_1 = -\dfrac{1}{(0+1)^2} a_0 = -a_0 = -1$

$a_2 = \dfrac{1-2}{2(1+1)^2} a_1 = -\dfrac{1}{8} \cdot (-a_0) = \dfrac{1}{8} a_0 = \dfrac{1}{8}$

$$y(x) = 1 - x + \frac{1}{8} x^2$$

Check: $\left.\begin{array}{l} y'(x) = -1 + \frac{1}{4}x \\ y''(x) = \frac{1}{4} \end{array}\right\} \Rightarrow 2x \cdot \frac{1}{4} + (2-x)\left(-1 + \frac{1}{4}x\right) + 2\left(1 - x + \frac{1}{8}x^2\right) =$

$\cancel{2} + \frac{1}{2}x + x - \frac{1}{4}x^2 + \cancel{2} - 2x + \frac{1}{4}x^2 = 0$ OK

c) Find suitable factor to put the equation on self-adjoint form. Identify the weight function $w(x)$

From A&W 10.7 we have the factor $\frac{1}{p_0(x)} \exp\left[\int^x \frac{p_1(t)}{p_0(t)} dt\right]$

where $p_0(x) = 2x$ and $p_1(x) = 2-x$

The factor becomes $\frac{1}{2x} \exp\left\{\int^x \frac{2-t}{2t} dt\right\} =$

$$= \frac{1}{2x} \exp\left\{\int^x \frac{dt}{t} - \int^x \frac{dt}{2}\right\} = \frac{1}{2x} \exp\left[\ln x - \frac{x}{2}\right] =$$

$$= \frac{1}{2x} \cdot \exp[\ln x] \cdot \exp\left[-\frac{x}{2}\right] = \frac{1}{2} e^{-x/2}$$

Verify: $\frac{1}{2} e^{-x/2} \cdot \{2xy'' + (2-x)y' + \lambda y\} =$

$$= x e^{-x/2} y'' + (1 - \tfrac{1}{2}x) e^{-x/2} y' + \tfrac{1}{2}\lambda e^{-x/2} y =$$

$$= \frac{d}{dx}\left\{x e^{-x/2} \frac{dy}{dx}\right\} + \frac{1}{2}\lambda e^{-x/2} y = 0$$

$$\underline{\underline{w(x) \equiv \frac{1}{2} e^{-x/2}}}$$

d) λ integer ≥ 0

$$\int_0^\infty y_m(x)\, y_n(x)\, w(x)\, dx = \begin{cases} 0, & m \neq n \\ >0, & m = n \end{cases}$$

To normalize the functions $y_n(x)$ we compute

$$\int_0^\infty [y_n(x)]^2 w(x)\, dx = \frac{1}{2}\int_0^\infty [y_n(x)]^2 e^{-x/2} dx \equiv N_n^2$$

~~Normalize the functions $y_n(x)$ as~~ $\cancel{\frac{y_n(x)}{N_n}}$

Then, given the expansion $f(x) = \sum_{n=0}^\infty A_n y_n(x)$

we have the coefficients A_n as

$$\int_0^\infty f(x)\, y_n(x)\, w(x)\, dx = \sum_{m=0}^\infty A_m \int_0^\infty y_n(x)\, y_m(x)\, w(x)\, dx =$$

$$= \sum_{m=0}^\infty A_m \cdot N_m^2\, \delta_{nm} = A_n N_n^2$$

$$\therefore A_n = \frac{1}{N_n^2}\int_0^\infty f(x)\, y_n(x)\, w(x)\, dx$$

with $N_n^2 = \frac{1}{2}\int_0^\infty [y_n(x)]^2 e^{-x/2} dx$

* The problem did say that $y_n(x)$ were orthonormal so assuming $N_n = 1$ was accepted

5. $L\frac{dI}{dt} + RI = E(t) \qquad I(0) = 0$

Laplace transform: $sL\tilde{I} + R\tilde{I} = \mathcal{L}\{E(t)\}$

$$\Rightarrow \tilde{I} = \frac{\mathcal{L}\{E(t)\}}{sL+R} = \frac{1}{L}\frac{\mathcal{L}\{E(t)\}}{s+\frac{R}{L}}$$

$E(t)$ is given by

a) $E_0 u(t)$: $\tilde{I} = E_0 \frac{\tilde{u}}{sL+R} = \frac{E_0}{L}\cdot\frac{\tilde{u}}{s+\frac{R}{L}}$

$$\frac{1}{s+\frac{R}{L}} = \mathcal{L}\{e^{-\frac{Rt}{L}}\}$$

Convolution: $I(t) = \frac{E_0}{L}\int_0^t e^{-\frac{Rz}{L}} u(t-z)\,dz$

b) $E_0\delta(t)$: $\tilde{I} = \frac{E_0}{L}\frac{1}{s+\frac{R}{L}}\mathcal{L}\{\delta(t)\} = \frac{E_0}{L}\frac{1}{s+\frac{R}{L}}$

$$I(t) = \frac{E_0}{L}e^{-Rt/L}$$

c) $E_0\sin\omega t$: $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$

$$\tilde{I} = \frac{E_0}{L}\cdot\frac{1}{s+\frac{R}{L}}\cdot\frac{\omega}{s^2+\omega^2}$$

Partial fractions:

(set $\frac{R}{L} \equiv \alpha$) $\qquad \frac{1}{s+\alpha}\cdot\frac{\omega}{s^2+\omega^2} = \frac{a}{s+\alpha} + \frac{bs+c}{s^2+\omega^2}$

$$\Rightarrow a = \frac{\omega}{\omega^2+\alpha^2} = -b, \quad c = \frac{\alpha\omega}{\omega^2+\alpha^2}$$

$$\tilde{I} = \frac{E_0}{L}\cdot\frac{1}{s+\frac{R}{L}}\cdot\frac{\omega}{s^2+\omega^2} = \frac{E_0}{L}\cdot\frac{\omega}{\omega^2+\left(\frac{R}{L}\right)^2}\left\{\frac{1}{s+\frac{R}{L}} + \frac{\frac{R}{L}-s}{s^2+\omega^2}\right\}$$

$\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\} = e^{-Rt/L} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t$

$\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t \qquad I(t) = \frac{E_0}{L\left[\omega^2+\left(\frac{R}{L}\right)^2\right]}\left\{\omega e^{-Rt/L} + \frac{R}{L}\sin\omega t - \omega\cos\omega t\right.$

5c

alternative $\int_0^t \sin[\omega(t-z)]\, e^{-\frac{R}{L}z}\, dz = \int_0^t \{\sin\omega t \cos\omega z - \cos\omega t \sin\omega z\}\, e^{-\frac{R}{L}z}\, dz$

Convolution theorem

$$= \sin\omega t \int_0^t \cos\omega z\, e^{-\frac{R}{L}z}\, dz - \cos\omega t \int_0^t \sin\omega z\, e^{-\frac{R}{L}z}\, dz \; ;$$

$$\int_0^t \cos\omega z\, e^{-\frac{R}{L}z}\, dz = -\frac{L}{R}\left[\cos\omega z\, e^{-\frac{R}{L}z}\right]_0^t - \frac{\omega L}{R}\int_0^t \sin\omega z\, e^{-\frac{R}{L}z}\, dz =$$

$$= \frac{L}{R}\left(1 - \cos\omega t\, e^{-\frac{R}{L}t}\right) + \omega\left(\frac{L}{R}\right)^2\left[\sin\omega z\, e^{-\frac{R}{L}z}\right]_0^t - \frac{\omega^2 L^2}{R^2}\int_0^t \cos\omega z\, e$$

$$\left(1 + \frac{\omega^2 L^2}{R^2}\right)\int_0^t \cos\omega z\, e^{-\frac{R}{L}z}\, dz = \frac{L}{R}\left(1 - \cos\omega t\, e^{-\frac{R}{L}t}\right) + \frac{\omega L^2}{R^2}\sin\omega t\, e^{-\frac{R}{L}}$$

First term:

$$\sin\omega t \int_0^t \cos\omega z\, e^{-\frac{R}{L}z}\, dz = \frac{R^2}{R^2+\omega^2L^2}\cdot\frac{L}{R}\Big\{\sin\omega t - \sin\omega t\cos\omega t\, e^{-\frac{R}{L}t} +$$

$$+ \frac{\omega L}{R}\sin^2\omega t\, e^{-\frac{R}{L}t}\Big\}$$

Second term:

$$\int_0^t \sin\omega z\, e^{-\frac{R}{L}z}\, dz = \left[-\frac{L}{R}\sin\omega z\, e^{-\frac{R}{L}z}\right]_0^t + \frac{\omega L}{R}\int_0^t \cos\omega z\, e^{-\frac{R}{L}z}\, dz =$$

$$= -\frac{L}{R}\sin\omega t\, e^{-\frac{R}{L}t} - \omega\left(\frac{L}{R}\right)^2\left[\cos\omega z\, e^{-\frac{R}{L}z}\right]_0^t - \frac{\omega^2L^2}{R^2}\int_0^t \sin$$

$$\left(1+\frac{\omega^2L^2}{R^2}\right)\int_0^t \sin\omega z\, e^{-\frac{R}{L}z}\, dz = -\frac{L}{R}\sin\omega t\, e^{-\frac{R}{L}t} + \omega\left(\frac{L}{R}\right)^2\left(1 - \cos\omega t\, e^{-\frac{R}{L}}\right.$$

$$\cos\omega t\int_0^t \sin\omega z\, e^{-\frac{R}{L}z}\, dz = \frac{R^2}{R^2+\omega^2L^2}\cdot\frac{L}{R}\Big\{-\sin\omega t\cos\omega t\, e^{-\frac{R}{L}t} +$$

$$+ \frac{\omega L}{R}\left(\cos\omega t - \cos^2\omega t\, e^{-\frac{R}{L}t}\right)$$

5c alternative cont'd

$$\therefore \int_0 \sin[\omega(t-z)]\, e^{-\frac{R}{L}z}\, dz = \frac{LR}{R^2+\omega^2L^2}\Big\{ \sin\omega t - \cancel{\sin\omega t \cos\omega t\, e^{-\frac{R}{L}t}} +$$

$$+ \frac{\omega L}{R}\sin^2\omega t\, e^{-\frac{R}{L}t} + \cancel{\sin\omega t\cos\omega t\, e^{-\frac{R}{L}t}} -$$

$$- \frac{\omega L}{R}\cos\omega t + \frac{\omega L}{R}\cos^2\omega t\, e^{-\frac{R}{L}t} \Big\} =$$

$$= \frac{LR}{R^2+\omega^2L^2}\Big\{ \sin\omega t + \frac{\omega L}{R} e^{-\frac{R}{L}t} - \omega\frac{L}{R}\cos\omega t \Big\} =$$

$$= \frac{L^2}{R^2+\omega^2L^2}\Big\{ \frac{R}{L}\sin\omega t + \omega e^{-\frac{R}{L}t} - \omega\cos\omega t \Big\} =$$

$$= \frac{1}{\omega^2+\left(\frac{R}{L}\right)^2}\Big\{ \omega e^{-\frac{R}{L}t} + \frac{R}{L}\sin\omega t - \omega\cos\omega t \Big\}$$

Better use of convolution theorem through

$$\int_0^t \sin(\omega z)\, e^{-\frac{R}{L}(t-z)}\, dz = e^{-\frac{R}{L}t}\int_0^t \sin(\omega z)\, e^{Rz/L}\, dz =$$

$$= \frac{L}{R}\Big[\sin(\omega z)\, e^{Rz/L}\Big]_0^t - \frac{\omega L}{R}\int_0^t \cos(\omega z)\, e^{Rz/L}\, dz =$$

$$= \frac{L}{R}\sin(\omega t)e^{Rt/L} - \omega\left(\frac{L}{R}\right)^2\Big[\cos(\omega z)\, e^{Rz/L}\Big]_0^t - \frac{\omega^2L^2}{R^2}\int_0^t \sin(\omega z)e^{Rz/L}$$

$$\Rightarrow \left(1+\frac{\omega^2L^2}{R^2}\right)\int_0^t \sin(\omega z)\, e^{Rz/L}\, dz = \frac{L}{R}\sin(\omega t)e^{Rt/L} +$$

$$+ \omega\left(\frac{L}{R}\right)^2 - \omega\left(\frac{L}{R}\right)^2\cos(\omega t)\, e^{Rt/L}$$

$$\Rightarrow e^{-\frac{R}{L}t}\int_0^t \sin(\omega z)\, e^{Rz/L}\, dz = \frac{1}{1+\frac{\omega^2L^2}{R^2}}\Big\{ \frac{L}{R}\sin(\omega t) + \omega\left(\frac{L}{R}\right)^2 e^{-\frac{R}{L}t} -$$

$$- \omega\left(\frac{L}{R}\right)^2\cos\omega t \Big\} =$$

$$= \frac{1}{\omega^2+\left(\frac{R}{L}\right)^2}\Big\{ \omega e^{-\frac{R}{L}t} + \frac{R}{L}\sin\omega t - \omega\cos\omega t \Big\}$$

6. $\frac{d}{dx}\left(x\frac{d\psi}{dx}\right) - \frac{n^2}{x}\psi = -f(x)$, $0 \le x \le 1$, $\psi(0)$ finite
$\psi(1) = 0$

Homogeneous equation: $\frac{d}{dx}\left(x\frac{dG}{dx}\right) - \frac{n^2}{x}G = 0$

Ansatz: $G = x^{\alpha}$ $\quad \frac{d}{dx}\left(x\frac{dx^{\alpha}}{dx}\right) - n^2 x^{\alpha-1} = 0$

$$(\alpha^2 - n^2)x^{\alpha-1} = 0 \Rightarrow \alpha = \pm n$$

$$\Rightarrow G(x) = b_1 x^n + b_2 x^{-n}$$

$$G(x,t) = \begin{cases} c_1 x^n, & 0 \le x < t \quad \text{finite at } x=0 \\ c_2(x^n - x^{-n}), & t < x \le 1 \quad G(1,t) = 0 \end{cases}$$

Continuity at $x=t$: $\quad c_1 t^n = c_2(t^n - t^{-n}) \quad (1)$

Derivative discontinuity at $x=t$: $\quad c_2(nt^{n-1} + nt^{-n-1}) - nc_1 t^{n-1} = -\frac{1}{t} \quad (2)$

$(1) \Rightarrow c_1 = c_2\{1 - t^{-2n}\}$

into (2) $\Rightarrow$: $c_2(\underbrace{\cancel{nt^{n-1}} + nt^{-n-1} - \cancel{nt^{n-1}} + nt^{-n-1}}_{2nt^{-n-1}}) = -\frac{1}{t}$

$$c_2 = -\frac{1}{2n}t^n$$

$$c_1 = -\frac{1}{2n}t^n(1 - t^{-2n}) = -\frac{1}{2n}(t^n - t^{-n})$$

$$\Rightarrow G(x,t) = \begin{cases} -\frac{1}{2n}\left\{(xt)^n - \left(\frac{x}{t}\right)^n\right\}, & 0 \le x < t \\ -\frac{1}{2n}\left\{(xt)^n - \left(\frac{t}{x}\right)^n\right\}, & t < x \le 1 \end{cases}$$

$$= \begin{cases} \frac{1}{2n}\left\{\left(\frac{x}{t}\right)^n - (xt)^n\right\}, & 0 \le x < t \\ \frac{1}{2n}\left\{\left(\frac{t}{x}\right)^n - (xt)^n\right\}, & t < x \le 1 \end{cases}$$

7. a) Equation is separable through the ansatz $u(r,t) = R(r)T(t)$

$$\frac{1}{c^2} R T'' = R''T + \frac{1}{r} R'T$$

$$\Rightarrow \frac{1}{c^2}\frac{T''}{T} = \frac{R'' + \frac{1}{r}R'}{R} = -\lambda^2$$

Equation for the time dependence: $T'' + (\lambda c)^2 T = 0$

$$\Rightarrow T(t) = A\cos(\lambda c t) + B\sin(\lambda c t)$$

For r-dependence $\quad R'' + \frac{1}{r}R' + \lambda^2 R = 0 \quad$ scale by r^2

$$\Rightarrow r^2 R'' + rR' + \lambda^2 r^2 R = 0 \quad \text{Bessel's equation zeroth order}$$

set $\varrho \equiv \lambda r$

and $y(\varrho) = R(\lambda r)$

$$\Rightarrow \varrho^2 y'' + \varrho y' + \varrho^2 y = 0$$

and $y(\varrho) = c\, J_0(\varrho) + d\, N_0(\varrho)$

Solutions have to be regular at $\varrho = 0 \Rightarrow d = 0$

$$\Rightarrow R(r) = J_0(\varrho) \equiv J_0(\lambda r)$$

$u(a,t) = 0 \Rightarrow \lambda = \frac{\alpha_m}{a}$ and $R_m(r) = J_0\left(\alpha_m \frac{r}{a}\right)$

Superposition gives general solution:

$$u(r,t) = \sum_{n=1}^{\infty} J_0\left(\alpha_n \frac{r}{a}\right)\left(A_n \cos\left(\frac{c\alpha_n}{a} t\right) + B_n \sin\left(\frac{c\alpha_n}{a} t\right)\right)$$

7 b) For $a = 2$, $c = 2$ and initial conditions

$$u(r,0) = 5 J_0\left(\frac{\alpha_3 r}{2}\right) \quad \text{and} \quad \frac{\partial u}{\partial t}(r,0) = 4\alpha_7 J_0\left(\frac{\alpha_7 r}{2}\right)$$

From a) : $u(r,0) = \sum_{n=1}^{\infty} J_0\left(\alpha_n \frac{r}{2}\right) A_n$

$$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty} J_0\left(\alpha_n \frac{r}{2}\right) \frac{2\alpha_n}{2} B_n = \sum_{n=1}^{\infty} J_0\left(\frac{\alpha_n r}{2}\right) \alpha_n B_n$$

Compare with $u(r,0) = 5 J_0\left(\frac{\alpha_3 r}{2}\right) \Rightarrow A_n = 5\delta_{3,n}$

$$\frac{\partial u}{\partial t}(r,0) = 4\alpha_7 J_0\left(\alpha_7 \frac{r}{2}\right) \Rightarrow B_n = 4\delta_{7,n}$$

so $u(r,t) = 5 J_0\left(\alpha_3 \frac{r}{2}\right) \cos(\alpha_3 t) + 4 J_0\left(\alpha_7 \frac{r}{2}\right) \sin(\alpha_7 t)$