Written Examination for Mathematical Methods of Physics
2010.01.04 at 09:00-15:00 Lösningar/solutions

1. $\int_0^{2\pi} \frac{d\vartheta}{13+5\sin\vartheta}$; substitute $z = e^{i\vartheta}$, $dz = ie^{i\vartheta}d\vartheta$

$d\vartheta = -i\frac{dz}{z}$; $\sin\vartheta = \frac{1}{2i}(z - \frac{1}{z})$

$$\int_0^{2\pi} \frac{d\vartheta}{13+5\sin\vartheta} = -i\oint \frac{dz}{z\{13+\frac{5}{2i}(z-\frac{1}{z})\}} = -i\oint \frac{dz}{13z+\frac{5}{2i}z^2-\frac{5}{2i}} =$$

$$= 2\oint \frac{dz}{26iz+5z^2-5}$$

Search for poles: $5z^2+26iz-5=0$

$$z^2+\frac{26i}{5}z-1=0$$

$$\left(z+\frac{13i}{5}\right)^2 = \left(\frac{13i}{5}\right)^2+1 = -\frac{169}{25}+1 = -\frac{144}{25}$$

$$\therefore z = -\frac{13i}{5} \pm \frac{12i}{5} = \begin{cases} -\frac{i}{5} \\ -5i \end{cases}$$

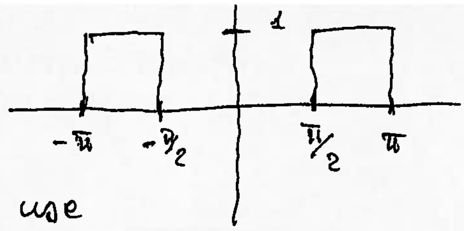
Only $z_+ = -\frac{i}{5}$ within the unit circle

$$\text{Res}\{z=z_+\} = \lim_{z\to-\frac{i}{5}} \frac{(z+\frac{i}{5})}{(z+\frac{i}{5})(z+5i)} = \frac{1}{5i-\frac{i}{5}} = \frac{5}{24i}$$

$$\therefore \int_0^{2\pi} \frac{d\vartheta}{13+5\sin\vartheta} = \frac{2}{\cancel{4}5}\oint \frac{dz}{26iz+5z^2-5} = \frac{2\cdot 2\pi i}{5}\text{Res}\{z=-\tfrac{i}{5}\} = \frac{4\pi i\cdot\cancel{5}}{24i} =$$

$$= \frac{\cancel{5}\pi}{6} \equiv \frac{\pi}{6}$$

2. $f(x) = \begin{cases} 1, & \pi/2 < |x| < \pi \\ 0, & \text{otherwise} \end{cases}$



The function is even so we can use the cosine transform

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos\omega t\,dt = \sqrt{\frac{2}{\pi}}\int_{\pi/2}^{\pi} \cos\omega t\,dt =$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{\omega}\sin\omega t\right]_{\pi/2}^{\pi} = \frac{1}{\omega}\sqrt{\frac{2}{\pi}}\left\{-\sin\frac{\omega\pi}{2} + \sin\omega\pi\right\}$$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty g_c(\omega)\cos\omega x\,d\omega = -\frac{2}{\pi}\int_0^\infty \cos\omega x \cdot \left\{\frac{\sin\omega\frac{\pi}{2} - \sin\omega\pi}{\omega}\right\} d\omega$$

3. Expand $f(x) = x^4$ on $-1 < x < 1$ using $p_\ell(x) = \sqrt{\frac{2\ell+1}{2}}\, P_\ell(x)$

Highest $\ell = 4$; $P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$

$$\Rightarrow \quad 8P_4 = 35x^4 - 30x^2 + 3 \quad \Rightarrow \quad x^4 = \frac{1}{35}\left(8P_4 + 30x^2 - 3\right) =$$

$$= \frac{8}{35}P_4 + \frac{6}{7}x^2 - \frac{3}{35}$$

Highest remaining $\ell = 2$:

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow 3x^2 = 2P_2 + 1 \Rightarrow x^2 = \frac{2}{3}P_2 + \frac{1}{3}$$

$$\Rightarrow x^4 = \frac{8}{35}P_4 + \frac{6}{7}\left(\frac{2}{3}P_2 + \frac{1}{3}\right) - \frac{3}{35} = \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{2}{7} - \frac{3}{35} =$$

$$= \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{10}{35} - \frac{3}{35} = \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}P_0$$

Convert to normalized $p_\ell(x) = \sqrt{\frac{2\ell+1}{2}}\, P_\ell(x)$

$$P_\ell(x) = \sqrt{\frac{2}{2\ell+1}}\, p_\ell(x) \Rightarrow P_4(x) = \sqrt{\frac{2}{9}}\, p_4(x) = \frac{\sqrt{2}}{3}\, p_4(x)$$

$$P_2(x) = \sqrt{\frac{2}{5}}\, p_2(x)$$

$$P_0(x) = \sqrt{2}\, p_0(x)$$

3 contd $x^4 = \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x) =$

$$= \frac{8\sqrt{2}}{105} P_4(x) + \frac{4}{7}\sqrt{\frac{2}{5}} P_2(x) + \frac{\sqrt{2}}{5} \bar{P}_0(x)$$

4. $y'' + 2y' + 2y = 10e^{x} + 6e^{-x}\cos x$

Homogeneous equation: $y'' + 2y' + 2y = 0$

Ansatz: $y(x) = \lambda e^{\lambda x} \Rightarrow \lambda^2 + 2\lambda + 2 = 0$

$$(\lambda+1)^2 = -1$$

$$\Rightarrow \lambda = -1 \pm i \equiv \begin{cases} \lambda_+ = \alpha \\ \lambda_- = \beta \end{cases}$$

Wronskian

$$\alpha = -1 + i$$

$$\beta = -1 - i$$

$$\begin{vmatrix} e^{\alpha x} & e^{\beta x} \\ \alpha e^{\alpha x} & \beta e^{\beta x} \end{vmatrix} = \beta e^{(\alpha+\beta)x} - \alpha e^{(\alpha+\beta)x} =$$

$$= (-1-i)e^{-2x} - (-1+i)e^{-2x} = -2i\, e^{-2x}$$

Particular solution giving $10e^{x}$

$$y_{p_1} = -e^{\beta x} \int^{x} \frac{e^{\alpha s}\, 10 e^{s}\, ds}{2i\, e^{-2s}} + e^{\alpha x} \int^{x} \frac{e^{\beta s}\, 10e^{s} ds}{2i\, e^{-2s}} =$$

$$= -5i \left\{ e^{\alpha x} \int^{x} e^{\beta s} e^{3s} ds - e^{\beta x} \int^{x} e^{\alpha s} e^{3s} ds \right\} =$$

$$= -5i \left\{ e^{\alpha x} \left[\frac{1}{(\beta+3)} e^{(\beta+3)s}\right]^{x} - e^{\beta x} \left[\frac{1}{(\alpha+3)} e^{(\alpha+3)s}\right]^{x} \right\} =$$

$\alpha+\beta = -2$

$\alpha - \beta = 2i$

$$= -5i \left\{ \frac{e^{(\alpha+\beta+3)x}}{\beta+3} - \frac{e^{(\alpha+\beta+3)x}}{\alpha+3} \right\} = -5i\, e^{x} \left\{ \frac{\alpha+3 - \beta+3}{(\alpha+3)(\beta+3)} \right\} =$$

$$= \frac{-5i \cdot 2i\, e^{x}}{(\alpha+3)(\beta+3)} = +2e^{x}$$ solves the equation

$$(\alpha+3)(\beta+3) = (\alpha+3)(\alpha^*+3) =$$

$$= |\alpha|^2 + 3(\alpha+\alpha^*) + 9 = 2 - 6 + 9 = 5$$

4 contd: $y_{P_2} = -3i\left\{ e^{\alpha x}\int^x e^{(\beta+2)s} e^{-s}\cos(s)\,ds - e^{\beta x}\int^x e^{(\alpha+2)s} e^{-s}\cos(s)\,ds \right\} =$

$$= -3i\left\{ e^{\alpha x}\int^x e^{(\beta+1)s}\cos(s)\,ds - e^{\beta x}\int^x e^{(\alpha+1)s}\cos(s)\,ds \right\}$$

integral $\int^x e^{\gamma s}\cos(s)\,ds = \frac{1}{2}\int^x e^{\gamma s}\left(e^{is}+e^{-is}\right)ds =$

$$= \frac{1}{2}\left\{ \int^x e^{(\gamma+i)s}\,ds + \int^x e^{(\gamma-i)s}\,ds \right\};$$

$$\cancel{= \frac{1}{2}\left\{ \frac{e^{(\gamma+i)x}}{\gamma+i} + \frac{e^{(\gamma-i)x}}{\gamma-i} \right\}}$$

1. $\gamma = \beta + 1 = -1 - i + 1 = -i$

$$\int^x e^{(\gamma+i)s}\,ds = \int^x ds = x$$

$$\int^x e^{(\gamma-i)s}\,ds = \int^x e^{-2is}\,ds = -\frac{1}{2i}e^{-2ix}$$

2. $\gamma = \alpha + 1 = -1 + i + 1 = i$

$$\int^x e^{(\gamma+i)s}\,ds = \int^x e^{2is}\,ds = \frac{1}{2i}e^{2ix}$$

$$\int^x e^{(\gamma-i)s}\,ds = \int^x ds = x$$

$$\Rightarrow y_{P_2} = -\frac{3i}{2}\left\{ e^{\alpha x}\left(x - \frac{1}{2i}e^{-2ix}\right) - e^{\beta x}\left(\frac{1}{2i}e^{2ix} + x\right) \right\} =$$

$$= -\frac{3i}{2}\left\{ x\left[e^{\alpha x} - e^{\beta x}\right] - \frac{1}{2i}\left\{ e^{(\alpha-2i)x} + e^{(\beta+2i)x} \right\} \right\} =$$

$$= 3xe^{-x}\frac{1}{2i}\left(e^{+ix} - e^{-ix}\right) + \frac{3}{2}\cdot\frac{1}{2}\left\{ e^{(-1-i)x} + e^{(-1+i)x} \right\} =$$

$$= 3xe^{-x}\sin x + \frac{3}{2}e^{-x}\cos x$$

solution?: $y' = 3e^{-x}\sin x - 3xe^{-x}\sin x + 3xe^{-x}\cos x - \frac{3}{2}e^{-x}\cos x - \frac{3}{2}e^{-x}\sin$

$y'' = -3e^{-x}\sin x + 3e^{-x}\cos x - 3e^{-x}\sin x + \cancel{3xe^{-x}\sin x} - 3xe^{-x}\cos x +$
$+ 3e^{-x}\cos x - 3xe^{-x}\cos x - \cancel{3xe^{-x}\sin x} + \frac{3}{2}e^{-x}\cos x + \frac{3}{2}e^{-x}\sin x +$

4 contd 2 :

$$+\frac{3}{2}e^{-x}\sin x - \frac{3}{2}e^{-x}\cos x =$$

$$= -3e^{-x}\sin x + 6e^{-x}\cos x - 6xe^{-x}\cos x$$

$$y' = \frac{3}{2}e^{-x}\sin x - 3xe^{-x}\sin x + 3xe^{-x}\cos x - \frac{3}{2}e^{-x}\cos x$$

$$y'' + 2y' + 2y = -3\cancel{e^{-x}}\sin x + 6e^{-x}\cos x - 6\cancel{xe^{-x}}\cos x +$$
$$+ 3\cancel{e^{-x}}\sin x - 3\cancel{e^{-x}}\cos x + 6\cancel{xe^{-x}}\cos x - 6\cancel{xe^{-x}}\sin x +$$
$$+ \cancel{3e^{-x}}\cos x + 6\cancel{xe^{-x}}\sin x = 6e^{-x}\cos x \quad \text{OK}$$

$$\therefore \; y(x) = e^{-x}(Ae^{ix} + Be^{-ix}) + 3xe^{-x}\sin x + \frac{3}{2}e^{-x}\cos x + 2e^{x}$$

5. $y'' + y = f(x) \qquad y(0) = y(\pi/2) = 0$

search for solution to $G''(x,x') + G(x,x') = \delta(x-x')$

with $G(0,x') = G(\pi/2, x') = 0$

The homogeneous equation is solved by

$G(x,x') = A(x')\sin x + B(x')\cos x$

so that
$$G(x,x') = \begin{cases} A(x')\sin x & 0 < x < x' < \pi/2 \\ B(x')\cos x & 0 < x' < x < \pi/2 \end{cases}$$

Continuity at $x = x'$: $\quad A(x')\sin x' = B(x')\cos x'$

Discontinuity of the derivative:

$$-B(x')\sin x' - A(x')\cos x' = 1$$

Solve for $A(x')$ and $B(x')$: $\quad A(x') = B(x')\dfrac{\cos x'}{\sin x'}$

$$-B(x')\sin x' - B(x')\frac{\cos^2 x'}{\sin x'} = 1$$

$$\Rightarrow B(x')(\sin^2 x' + \cos^2 x') = -\sin x'$$

$$\therefore B(x') = -\sin x', \quad A(x') = -\cos x'$$

$$\Rightarrow G(x,x') = \begin{cases} -\cos x'\sin x, & 0 < x < x' < \pi/2 \\ -\sin x'\cos x, & 0 < x' < x < \pi/2 \end{cases}$$

$$y(x) = \int_0^{\pi/2} G(x,x')f(x')dx' = -\cos x\int_0^x \sin x' \cdot f(x')dx' - \sin x\int_x^{\pi/2}\cos x' f(x')dx'$$

Check: $x \to 0$ second term zero ($\sin x \to 0$)
first — " — $\int_0^x dx' \to 0$

$x \to \frac{\pi}{2}$ first term $\cos x \to 0$
second term $\int_x^{\pi/2} dx' \to 0$

OK

6. Heat conduction equation

$$\frac{\partial u}{\partial t} = b^2 \nabla^2 u \quad , \quad u(r,\vartheta,t=0) = 100\, r \sin\vartheta$$

Transform to planar polar coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}$$

Make separation Ansatz: $u(r,\vartheta,t) = R(r)\Theta(\vartheta)T(t)$

$$\Rightarrow \frac{R\Theta T'}{b^2} = \cancel{b}\,\Theta T\left\{\frac{1}{r}\frac{d}{dr}(rR')\right\} + \frac{1}{r^2}RT\Theta''$$

$$\Rightarrow \frac{1}{b^2}\frac{T'}{T} = \cancel{b}\,\frac{\cancel{\Theta}}{R}\left\{\frac{1}{r}\frac{d}{dr}(rR')\right\} + \frac{1}{r^2}\frac{\Theta''}{\Theta}$$

Left hand side depends only on t, RHS only on r, ϑ

$$\Rightarrow \begin{cases} (1) & \frac{T'}{T} = -\lambda^2 b^2 \Rightarrow T' + b^2\lambda^2 T = 0 \Rightarrow T(t) = Ce^{-\lambda^2 t b^2} \\ (2) & \cancel{b}\,\frac{\cancel{\Theta}}{R}\left\{\frac{1}{r}\frac{d}{dr}(rR')\right\} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2 \end{cases}$$

Scale (2) by r^2 and separate $\frac{\cancel{b}\,r}{R}\frac{d}{dr}(rR') + \lambda^2 r^2 = -\frac{\Theta''}{\Theta} = +$

$$\Rightarrow \begin{cases} \Theta'' + n^2\Theta = 0 \Rightarrow \Theta(\vartheta) = A\cos(n\vartheta) + B\sin(n\vartheta) \\ \cancel{b}\,r\frac{d}{dr}(rR') + (\lambda^2 r^2 - n^2)R = 0 \quad \text{Bessel's equation} \end{cases}$$

Since the origin is included only $J_n(\lambda r)$

~~Aff~~ For $t>0$ $u(a,\vartheta,t) = 0 \Rightarrow J_n(\lambda a) = 0$

General solution $\Rightarrow$ Discrete set of $\lambda \to \lambda_n$

$$u(r,\vartheta,t) = \sum \sum J_n(\lambda_{n\nu} r)\{A_{n\lambda_n}\cos n\vartheta + B_{n\lambda_n}\sin n\vartheta\}e^{-\lambda_n^2 b^2 t}$$

$u(r,\vartheta,0) = 100\, r\sin\vartheta \Rightarrow A_{n\lambda_n} = 0 \quad , \quad n = 1$

~~and~~

Set $\lambda_{n\nu} = \frac{\alpha_{n\nu}}{a}$, $n = 1$

6 contd

$$u(r, \vartheta, t) = \sin\vartheta \sum_m B_{1m} J_1\left(\alpha_{1m}\frac{r}{a}\right) e^{-\left(\frac{\alpha_{1m}}{a}\right)^2 b^2 t}$$

Find B_{1m} such that $u(r, \vartheta, 0) = 100 r \sin\vartheta$

Use orthogonality and normalization

$$\int_0^a \left[J_\nu\left(\alpha_{\nu m}\frac{r}{a}\right)\right]^2 r\,dr = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu m})\right]^2 \quad \text{(Arfken 11.50)}$$

So $$100 r = \sum_{m=1}^{\infty} B_{1m} J_1\left(\alpha_{1m}\frac{r}{a}\right)$$

$$\Rightarrow B_{1m} = \frac{2 \cdot 100}{a^2\left[J_2(\alpha_{1m})\right]^2}\int_0^a r J_1\left(\alpha_{1m}\frac{r}{a}\right) r\,dr .$$

$$\int_0^a r^2 J_1\left(\alpha_{1m}\frac{r}{a}\right) dr = \left[\begin{array}{ll} \rho = \alpha_{1m}\frac{r}{a}, & r = \frac{a\rho}{\alpha_{1m}} \\ dr = \frac{a}{\alpha_{1m}} d\rho & \end{array}\right] =$$

$$= \left(\frac{a}{\alpha_{1m}}\right)^3 \int_0^{\alpha_{1m}} \rho^2 J_1(\rho)\,d\rho = \left(\frac{a}{\alpha_{1m}}\right)^3 \int_0^{\alpha_{1m}} \frac{d}{d\rho}\left[\rho^2 J_2(\rho)\right] d\rho =$$

$$= \frac{a^3}{\alpha_{1m}} J_2(\alpha_{1m})$$

Arfken 11.15:

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$$\therefore \quad B_{1m} = \frac{200\,a}{\alpha_{1m} J_2(\alpha_{1m})}$$

~~$\Rightarrow u(r, \vartheta, t) = \sin\vartheta \sum$~~

$$\Rightarrow u(r, \vartheta, t) = 200\,a \sin\vartheta \sum_{m=1}^{\infty} \frac{J_1\left(\alpha_{1m}\frac{r}{a}\right)}{\alpha_{1m} J_2(\alpha_{1m})} e^{-\left(\frac{b\alpha_{1m}}{a}\right)^2 t}$$

7. $J_n(x) = \sum_{s=0} \frac{(-1)^s}{s!\,(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$

Verify

a) $J_1(x) + J_3(x) = \frac{4}{x} J_2(x)$

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+1)!}\left(\frac{x}{2}\right)^{2s+1} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+3)!}\left(\frac{x}{2}\right)^{2s+3} =$$

$$= \frac{x}{2} + \sum_{s=1}^{\infty} \frac{(-1)^s}{s!\,(s+1)!}\left(\frac{x}{2}\right)^{2s+1} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+3)!}\left(\frac{x}{2}\right)^{2s+3} =$$

$$= \frac{x}{2} - \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+1)!\,(s+2)!}\left(\frac{x}{2}\right)^{2s+3} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+3)!}\left(\frac{x}{2}\right)^{2s+3} =$$

$$= \frac{x}{2} + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+2)!}\left(\frac{x}{2}\right)^{2s+3} \underbrace{\left\{\frac{1}{s+3} - \frac{1}{s+1}\right\}}_{-\frac{2}{(s+1)(s+3)}} =$$

$$= \frac{x}{2} + 2\sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{(s+1)!\,(s+3)!}\left(\frac{x}{2}\right)^{2s+3} =$$

$s=0: \; 2 \cdot \frac{1}{0!\,2!}\,\frac{x}{2} =$

$$= \frac{x}{2} + 2\sum_{s=1}^{\infty} \frac{(-1)^s}{s!\,(s+2)!}\left(\frac{x}{2}\right)^{2s+1} = 2\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+2)!}\left(\frac{x}{2}\right)^{2s+1} =$$

$$= \frac{4}{x}\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+2)!}\left(\frac{x}{2}\right)^{2s+2} = \frac{4}{x} J_2(x)$$

b) $\frac{d}{dx}\left(x J_1(x)\right) = x J_0(x)$

$$\frac{d}{dx}\left\{x J_1(x)\right\} = \frac{d}{dx}\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+1)!}\,\frac{x^{2s+2}}{2^{2s+1}} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+1)!}\,(2s+2)\,\frac{x^{2s+1}}{2^{2s+1}} =$$

$$= x\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+1)!}\,(s+1)\,\frac{x^{2s}}{2^{2s}} = x\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,s!}\left(\frac{x}{2}\right)^{2s} = x J_0(x)$$

8. $x^2 y'' + 4xy' + (x^2+2)y = 0$

Frobenius Ansatz: $y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+s}, \quad a_0 \neq 0$

$$y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2}$$

~~Equation~~ $\cancel{\sum_{n=0}^{\infty} \{ a_n (n+s)(n+s-1) \, x^{n+s} }$

We have the same power for x^2y'', $4xy'$ and $2y$:

$$\sum_{n=0}^{\infty} a_n x^{n+s} \{ (n+s)(n+s-1) + 4(n+s) + 2 \} + \sum_{m=0}^{\infty} a_m x^{m+s+2} = 0$$

Lowest power for $n=0$: $a_0 \{ s(s-1) + 4s + 2 \} = 0$

Indicial equation $s^2 + 3s + 2 = 0$ with roots $s = -1, -2$

$s = -1$: $\quad a_1$: $\quad a_1 \{ 0 + 4 \cdot 0 + 2 \} = 0 \quad \Rightarrow a_1 = 0$

shift the first sum $m \rightarrow m+2$

$$\Rightarrow a_{n+2} \{ \underbrace{(n+2-1)}_{n+1} \underbrace{(n+2-2)}_{n} + 4\underbrace{(n+2-1)}_{n+1} + 2 \} + a_n = 0$$

$$\Rightarrow a_{n+2} = - \frac{a_n}{(n+2)(n+3)}$$

odd powers $\rightarrow$ zero $\quad a_{2n+1} = 0$

$$a_2 = -\frac{a_0}{2 \cdot 3}, \quad a_4 = -\frac{a_2}{4 \cdot 5} = \frac{1}{5!} a_0, \quad a_6 = -\frac{a_4}{6 \cdot 7} = -\frac{1}{7!} a_0$$

$$\Rightarrow y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n-1} = \frac{a_0}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = a_0 \frac{\sin x}{x^2}$$

8 cont.

$s = -2$: $\quad a_1 \{ \underbrace{(1-2)(1-2-1) + 4(1-2) + 2}_{=0} \} = 0$

a_1 undetermined, set $a_1 = 0$

general $\quad a_{n+2} \{ (n+2-2)(n+2-2-1) + 4(n+2-2) + 2 \} + a_n = 0$

$$a_{n+2} \{ n(n-1) + 4n + 2 \} + a_n = 0$$

$$a_{n+2} = - \frac{a_n}{(n+1)(n+2)}$$

$$a_2 = - \frac{1}{1 \cdot 2} a_0 , \quad a_4 = - \frac{a_2}{3 \cdot 4} = \frac{1}{4!} a_0$$

$$a_{2n} = \frac{(-1)^n}{(2n)!} a_0$$

$$\Rightarrow \quad y(x) = \frac{a_0}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = a_0 \frac{\cos x}{x^2}$$

9. Show $\mathcal{L}^{-1}\{(s^2+a^2)^{-2}\} = \frac{1}{2a^3}\sin at - \frac{1}{2a^2}\, t\cos at$

The ~~derivative~~ transform of $\sin at$: $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

Alt. 1 Take derivative wrt a to generate denominator

$$\int_0^\infty e^{-st}\sin at\,dt = \frac{a}{s^2+a^2}$$

$$\frac{d}{da}\int_0^\infty e^{-st}\sin at\,dt = \int_0^\infty e^{-st}\, t\cos at\,dt = \frac{1}{s^2+a^2} - \frac{2a^2}{(s^2+a^2)^2}$$

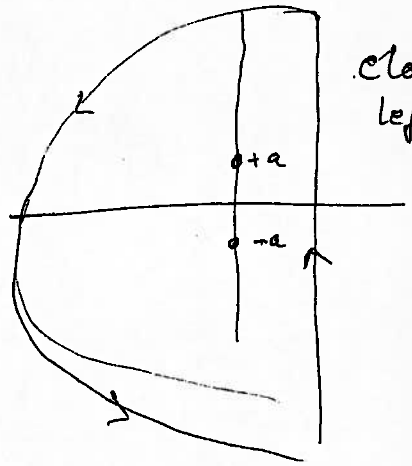
$$\therefore \frac{1}{(s^2+a^2)^2} = \frac{1}{2a^2}\cdot\frac{1}{s^2+a^2} - \frac{1}{2a^2}\int_0^\infty e^{-st}\, t\cos at\,dt =$$

$$= \frac{1}{2a^3}\,\frac{a}{s^2+a^2} - \frac{1}{2a^2}\int_0^\infty e^{-st}\, t\cos at\,dt =$$

$$= \frac{1}{2a^3}\int_0^\infty e^{-st}\sin at\,dt - \frac{1}{2a^2}\int_0^\infty e^{-st}\, t\cos at\,dt$$

$$\therefore \frac{1}{(s^2+a^2)^2} = \int_0^\infty e^{-st}\left\{\frac{1}{2a^3}\sin at - \frac{1}{2a^2}\, t\cos at\right\}dt$$

Alt 2 Bromwich integral



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$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}}{(s^2+a^2)^2}\,ds = \sum\{\text{Residues}\}$$

$$\frac{e^{st}}{(s^2+a^2)^2} = \frac{e^{st}}{(s+ia)^2(s-ia)^2}$$ Double poles at $z = \pm ia$

$$\text{Res}\{z=ia\} = \frac{d}{ds}\,\frac{(s-ia)^2 e^{st}}{(s+ia)^2(s-ia)^2}\bigg|_{s=ia} =$$

$$= -\frac{2e^{st}}{(s+ia)^3}\bigg|_{s=ia} + \frac{te^{st}}{(s+ia)^2}\bigg|_{s=ia} = -\frac{2e^{iat}}{(2ia)^3} + \frac{te^{iat}}{(2ia)^2} = \frac{1}{4}\,\frac{e^{iat}}{ia^3} - \frac{te^{iat}}{4a^2}$$

$$\text{Res}\{z=-ia\} = -\frac{2e^{st}}{(s-ia)^3}\bigg|_{s=-ia} + \frac{te^{st}}{(s-ia)^2}\bigg|_{s=-ia} = -\frac{2e^{-iat}}{(-2ia)^3} + \frac{te^{-iat}}{(-2ia)^2} = -\frac{1}{4}\,\frac{e^{-iat}}{ia^3} - \frac{te^{-iat}}{4a^2}$$

$$\Rightarrow \mathcal{L}^{-1}\{(s^2+a^2)^{-2}\} = \frac{1}{2a^3}\left\{\frac{e^{iat}-e^{-iat}}{2i}\right\} - \frac{t}{2a^2}\left\{\frac{e^{iat}+e^{-iat}}{2}\right\} = \frac{1}{2a^3}\sin at - \frac{t}{2a^2}\cos a$$

10. Solve $y'' + 3y' + 2y = e^{-t}$ using Laplace transform & convol

$y(0) = y'(0) = 0$, $y(t) = 0$ for $t < 0$

Transform equation $s^2\tilde{y} + 3s\tilde{y} + 2\tilde{y} = \mathcal{L}\{e^{-t}\}$

$$\tilde{y} = \frac{\mathcal{L}\{e^{-t}\}}{s^2 + 3s + 2}$$

Find roots: $s^2 + 3s + 2 = 0$

$$\left(s + \frac{3}{2}\right)^2 = \frac{9}{4} - \frac{8}{4} = \frac{1}{4}$$

$$s = -\frac{3}{2} \pm \frac{1}{2} = \begin{cases} -1 \\ -2 \end{cases}$$

Partial fractions: $\dfrac{1}{s^2 + 3s + 2} = \dfrac{1}{(s+1)(s+2)} = \dfrac{a}{s+1} + \dfrac{b}{s+2} = \dfrac{as + 2a + bs +}{(s+1)(s+2)}$

$a + b = 0 \Rightarrow a = -b$

$2a + b = 1 \Rightarrow b = -1$

$a = 1$

Arfken eq. 15.196

$$\Rightarrow \tilde{y} = \mathcal{L}\{e^{-t}\}\left\{\frac{1}{s+1} - \frac{1}{s+3}\right\} = \underbrace{\mathcal{L}\{e^{-t}\}}_{h(t)}\mathcal{L}\{\underbrace{e^{-t} - e^{-2t}}_{g(t)}\} = H(s)G(s$$

$$\int_0^\infty e^{-st}e^{-\alpha t}dt = \int_0^\infty e^{-(s+\alpha)t}dt = \frac{1}{s+\alpha}$$

Convolution theorem: $y(t) = \int_0^t g(\tau)h(t-\tau)d\tau =$

$$= \int_0^t (e^{-\tau} - e^{-2\tau})e^{-(t-\tau)}d\tau = e^{-t}\int_0^t (1 - e^{-\tau})d\tau =$$

$$= e^{-t}\left\{t + \left[e^{-\tau}\right]_0^t\right\} = e^{-t}(t + e^{-t} - 1) = te^{-t} + e^{-2t} - e^{-t}$$

Check: $y' = e^{-t} - te^{-t} - 2e^{-2t} + e^{-t} = 2e^{-t} - 2e^{-2t} - te^{-t}$

$y'' = -2e^{-t} + 4e^{-2t} - e^{-t} + te^{-t} = -3e^{-t} + 4e^{-2t} + te^{-t}$

$$y'' + 3y' + 2y = -3e^{-t} + 4e^{-2t} + te^{-t} +$$
$$+ 6e^{-t} - 6e^{-2t} - 3te^{-t} +$$
$$+ 2e^{-t} + 2e^{-2t} + 2te^{-t} = e^{-t}$$ OK