1. Separate the integrand into partial fractions and show by explicitly evaluating the resulting integrals that

$$\oint \frac{dz}{z^2+z} = 0 \text{ where the closed contour is a circle defined by } |z| = R > 1 \qquad (2p)$$

1. $\frac{1}{z^2+z} = \frac{1}{z(z+1)} = \frac{a}{z} + \frac{b}{z+1} = \frac{az+a+bz}{z(z+1)} \Rightarrow a=1,\ b=-1$

$$\oint_C \frac{dz}{z^2+z} = \oint_C \frac{dz}{z} - \oint_C \frac{dz}{z+1} = I_1 - I_2$$

I_1: $z = Re^{i\vartheta}$, $dz = iRe^{i\vartheta}d\vartheta \Rightarrow \frac{dz}{z} = i\,d\vartheta \Rightarrow I_1 = i\int_0^{2\pi} d\vartheta = 2\pi i$

I_2: $z+1 = Re^{i\vartheta}$, " — $\Rightarrow \frac{dz}{z+1} = i\,d\vartheta \Rightarrow I_2 = i\int_0^{2\pi} d\vartheta = 2\pi i$

$\therefore \oint_C \frac{dz}{z^2+z} = 2\pi i - 2\pi i = 0$

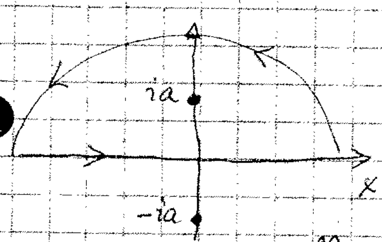
2. Evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx$ where $k>0$. (3p)

2. $$\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx + \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2} dx$$ poles at $x = \pm ia$

$$= \frac{1}{2}(I_1 + I_2)$$

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx \; ; \quad k>0$$ Use Jordan's lemma to close in UHP

The half circle does not contribute



$$I_1 = 2\pi i \, Res\{z = ia\} =$$

$$= 2\pi i \lim_{z \to ia} \frac{(z-ia)e^{ikz}}{(z-ia)(z+ia)} = \frac{2\pi i \, e^{-ka}}{2ia} = \frac{\pi}{a} e^{-ka}$$

$$I_2 = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2} dx \; ;$$ Close in lower half plane, pole encircled clockwise

$$I_2 = -2\pi i \, Res\{z = -ia\} = -2\pi i \lim_{z \to -ia} \frac{(z+ia)e^{-ikz}}{(z-ia)(z+ia)} =$$

$$= -\frac{2\pi i}{-2ia} e^{-ik(-ia)} = \frac{\pi}{a} e^{-ka}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx = \underline{\underline{\frac{\pi}{a} e^{-ka}}}$$

Alternative: $$\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx = Re \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2} dx$$ and continue as above

3. Show by direct differentiation that $J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+\nu)!}\left(\frac{x}{2}\right)^{\nu+2s}$ satisfies the Bessel's differential equation

$$x^2 J_\nu''(x) + xJ_\nu'(x) + (x^2 - \nu^2)J_\nu(x) = 0 \qquad \text{(3p)}$$

3. $$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+\nu)!}\left(\frac{x}{2}\right)^{\nu+2s} \equiv \sum_{s=0}^{\infty} C_s \left(\frac{x}{2}\right)^{\nu+2s}$$

$$J_\nu'(x) = \frac{1}{2}\sum_{s=0}^{\infty} C_s(\nu+2s)\left(\frac{x}{2}\right)^{\nu+2s-1} = \frac{1}{x}\sum_{s=0}^{\infty} C_s\cdot(\nu+2s)\left(\frac{x}{2}\right)^{\nu+2s}$$

$$J_\nu''(x) = \frac{1}{4}\sum_{s=0}^{\infty} C_s\cdot(\nu+2s)(\nu+2s-1)\left(\frac{x}{2}\right)^{\nu+2s-2} = \frac{1}{x^2}\sum_{s=0}^{\infty} C_s\cdot(\nu+2s)(\nu+2s-1)\left(\frac{x}{2}\right)^{\nu+2s}$$

Bessel's D.E.: $x^2 J_\nu'' + xJ_\nu' + (x^2-\nu^2)J_\nu =$

$$= \sum_{s=0}^{\infty} C_s\left\{(\nu+2s)(\nu+2s-1) + \nu+2s - \nu^2\right\}\left(\frac{x}{2}\right)^{\nu+2s} +$$

$$+4\sum_{s=0}^{\infty} C_s\left(\frac{x}{2}\right)^{\nu+2s+2} = 4\sum_{s=0}^{\infty} C_s\cdot s(\nu+s)\left(\frac{x}{2}\right)^{\nu+2s} + 4\sum_{s=0}^{\infty} C_s\left(\frac{x}{2}\right)^{\nu+2(s+1)}$$

$$(\nu+2s)(\nu+2s-1) + \nu+2s - \nu^2 = \cancel{\nu^2} + 4\nu s - \cancel{\nu} + 4s^2 - \cancel{2s} + \cancel{\nu} + \cancel{2s} - \cancel{\nu^2} =$$
$$= 4s(\nu+s)$$

The first sum: $$4\sum_{s=0}^{\infty} C_s\cdot s(\nu+s)\left(\frac{x}{2}\right)^{\nu+2s} = 4\cdot C_s\cdot 0(\nu+0)\left(\frac{x}{2}\right)^{\nu} +$$

$$+4\sum_{s=1}^{\infty} \frac{(-1)^s s(\nu+s)}{s!(\nu+s)!}\left(\frac{x}{2}\right)^{\nu+2s} = 4\sum_{s=0}^{\infty} \frac{(-1)^{s+1}(s+1)(\nu+s+1)}{(s+1)!(\nu+s+1)!}\left(\frac{x}{2}\right)^{\nu+2s+2}$$

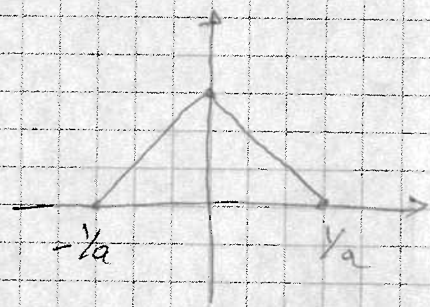
$$= -4\sum_{s=0}^{\infty} \frac{(-1)^s}{s!(\nu+s)!}\left(\frac{x}{2}\right)^{\nu+2(s+1)} = -4\sum_{s=0}^{\infty} C_s\left(\frac{x}{2}\right)^{\nu+2(s+1)}$$

$$\therefore\ x^2 J_\nu'' + xJ_\nu' + (x^2-\nu^2)J_\nu = 0$$

4. Find the Fourier transform of the triangular pulse (3p)

$$f(x)=\begin{cases} h(1-a|x|) & |x|\le 1/a \\ 0 & |x|>1/a \end{cases}$$

4. $f(x) = \begin{cases} h(1-a|x|) & |x| \le 1/a \\ 0 & |x| > 1/a \end{cases}$



$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-1/a}^{0} h(1+ax) e^{ikx} dx + \int_{0}^{1/a} h(1-ax) e^{ikx} dx \right\} =$$

$$= \frac{h}{\sqrt{2\pi}} \left\{ \left[\frac{e^{ikx}}{ik} \right]_{-1/a}^{0} + a \int_{-1/a}^{0} x e^{ikx} dx + \left[\frac{e^{ikx}}{ik} \right]_{0}^{1/a} - a \int_{0}^{1/a} x e^{ikx} dx \right\} =$$

$$= \frac{h}{\sqrt{2\pi}} \left\{ \underbrace{\frac{1}{ik} - \frac{e^{-ik/a}}{ik} + \frac{e^{ik/a}}{ik} - \frac{1}{ik}}_{2\frac{\sin(k/a)}{k}} + a \frac{d}{dk} \frac{1}{i} \underbrace{\int_{-1/a}^{0} e^{ikx} dx}_{\frac{1}{ik}(1-e^{-ik/a})} - a \frac{1}{i} \frac{d}{dk} \underbrace{\int_{0}^{1/a} e^{ikx} dx}_{\frac{1}{ik}(e^{ik/a}-1)} \right\}$$

$$= \frac{h}{\sqrt{2\pi}} \left\{ 2\frac{\sin(k/a)}{k} + \frac{a}{i} \left\{ -\frac{2}{ik^2} - \frac{d}{dk} \left(\frac{e^{ik/a} + e^{-ik/a}}{ik} \right) \right\} \right\} =$$

$$= \frac{h}{\sqrt{2\pi}} \left\{ \frac{2}{k} \sin\left(\frac{k}{a}\right) + \frac{2a}{k^2} + 2a \frac{d}{dk} \left[\frac{\cos(k/a)}{k} \right] \right\} =$$

$$= \frac{h}{\sqrt{2\pi}} \left\{ \frac{2}{k} \sin\left(\frac{k}{a}\right) + \frac{2a}{k^2} + 2a \left(\frac{-\frac{1}{a} \sin\left(\frac{k}{a}\right) \cdot k - \cos\left(\frac{k}{a}\right)}{k^2} \right) \right\} =$$

$$= \frac{h}{\sqrt{2\pi}} \left\{ \frac{2a}{k^2} - \frac{2a}{k^2} \cos\left(\frac{k}{a}\right) \right\} = \sqrt{\frac{2}{\pi}} \frac{ah}{k^2} \left(1 - \cos\left(\frac{k}{a}\right) \right)$$

5. Calculate the Legendre polynomial $P_4(x)$ at x=0.1 by using a forward recursion formula given the following initial values:

$P_0(0.1) = 1$

$P_1(0.1) = 0.1$ (2p)

5. Use $P_{n+1}(x) = 2x\,P_n(x) - P_{n-1}(x) - \frac{1}{n+1}\left[x\,P_n(x) - P_{n-1}(x)\right]$

$x = 0.1$ $P_2(0.1) = 2\cdot 0.1\cdot 0.1 - 1 - \frac{1}{2}\left[0.1\cdot 0.1 - 1\right] =$

$= 0.02 - 1 + \frac{1}{2}\cdot 0.99 = -0.485$

$P_3(0.1) = 2\cdot 0.1\cdot(-0.485) - 0.1 - \frac{1}{3}\left[0.1\cdot(-0.485) - 0.1\right] =$

$= -0.1475$

$P_4(0.1) = 2\cdot 0.1\,(-0.1475) - (-0.485) - \frac{1}{4}\left[0.1\cdot(-0.1475) - (-0.485)\right]$

$= 0.3379375$

6. Obtain the Frobenius power-series solution of the homogeneous differential equation

$$x(1-x)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$$ (3p)

6. $x(1-x)y'' - xy' - y = 0$

Power series expansion $y(x) = x^k \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda = \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+k}$, $a_0 \neq 0$

We see directly from the equation that $y(0) = 0$ so $k \neq 0$

$(0\cdot(1-0)y''(0) - 0\cdot y'(0) - y(0) = -y(0) = 0$)

$$y'(x) = \sum_{\lambda=0}^{\infty} a_\lambda (\lambda+k) x^{\lambda+k-1}$$

$$y''(x) = \sum_{\lambda=0}^{\infty} a_\lambda (\lambda+k)(\lambda+k-1) x^{\lambda+k-2}$$

The equation: $\sum_{\lambda=0}^{\infty} x(1-x) a_\lambda (\lambda+k)(\lambda+k-1) x^{\lambda+k-2} - \sum_{n=0}^{\infty} x a_n (n+k) x^{n+k-1} -$

$$-\sum_{m=0}^{\infty} a_m x^{m+k} = \sum_{\lambda=0}^{\infty} a_\lambda (\lambda+k)(\lambda+k-1) x^{\lambda+k-1} -$$

$$-\sum_{n=0}^{\infty} a_n \{(n+k)(n+k-1) + n+k + 1\} x^{n+k} = a_0 k(k-1)x^{k-1} + \sum_{\lambda=1}^{\infty} a_\lambda (\lambda+k)(\lambda+k-1) x^{\lambda+k-1}$$

$$-\sum_{n=0}^{\infty} a_n \{(n+k)^2 + 1\} x^{n+k} = a_0 k(k-1) x^{k-1} + \cancel{\sum_{n=0}^{\infty}}$$

$$+\sum_{n=0}^{\infty} \left\{ a_{n+1}(n+k)(n+k+1) - a_n\{(n+k)^2+1\} \right\} x^{n+k}$$

$k=0$ is incompatible with $a_0 \neq 0$

$k=1$: $a_{n+1}(n+1)(n+2) = a_n\{(n+1)^2+1\} \Rightarrow a_{n+1} = \dfrac{(n+1)^2+1}{(n+1)(n+2)} a_n$

Take $a_n = \dfrac{n^2+1}{n(n+1)} a_{n-1} =$

$$= \frac{n^2+1}{n(n+1)} \cdot \frac{(n-1)^2+1}{(n-1)\cdot n} a_{n-2} = \cdots = \prod_{m=1}^{n} \frac{m^2+1}{m(m+1)} a_0 = \frac{\prod_{m=1}^{n}(m^2+1)}{n!(n+1)!} a_0$$

7. An infinite string has an initial displacement $u_0(x)$ and an initial velocity $v(x) = u'(x,0) = v_0 \exp(-ax)$. Solve the wave equation

$\frac{\partial^2}{\partial x^2} u(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u(x,t)$ for the amplitude $u(x,t)$ at subsequent times. (4p)

7. $u(x,0) = u_0(x)$

$u'(x,0) = V_0 \exp(-ax)$

Solutions to the wave equation have the general form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t) = f(x-ct) + g(x+ct)$$

From the initial conditions we get

$$u(x,0) = u_0(x) \Rightarrow u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + f(x,t)$$

$f(x,t)$ is determined from the condition on the derivative

$$u'(x,0) = V_0 \exp(-ax) \Rightarrow u'(x,t) = \frac{1}{2}\left[V_0 \exp(-a(x-ct)) + V_0 \exp(-a(x+ct$$

$$\Rightarrow u(x,t) = \frac{V_0}{2} e^{-ax} \cdot \left\{ \int e^{act} dt + \int e^{-act} dt \right\} =$$

$$= \frac{V_0}{2} e^{-ax} \left\{ \frac{1}{ac} e^{act} - \frac{1}{ac} e^{-act} \right\} =$$

$$= \frac{V_0}{2ac} \left\{ e^{-a(x-ct)} - e^{-a(x+ct)} \right\}$$

$$\therefore\ u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{V_0}{2ac}\left[e^{-a(x-ct)} - e^{-a(x+ct)}\right]$$

8. Calculate the Green function for the differential equation $\left(\frac{d^2}{dx^2}+\frac{2}{x}\frac{d}{dx}-\frac{l(l+1)}{x^2}\right)y(x)=R(x)$ in the interval $(0,\infty)$ satisfying the boundary conditions $y(0)=y(\infty)=0$. Hint: A simple Ansatz solves the homogeneous equation. (4p)

8. Find the Green fctn for the equation

$$\left(\frac{d^2}{dx^2}+\frac{2}{x}\frac{d}{dx}-\frac{l(l+1)}{x^2}\right)y(x)=R(x) \qquad (0,\infty) \qquad y(0)=y(\infty)=0$$

G_1, G_2 solutions to homogeneous equation

Make the ansatz $y(x)=x^\alpha$

$$\Rightarrow \{\alpha(\alpha-1)+2\alpha-l(l+1)\}x^\alpha=0$$

$$\alpha^2+\alpha=l(l+1), \quad \left(\alpha+\frac{1}{2}\right)^2=l(l+1)+\frac{1}{4}$$

$$\alpha=-\frac{1}{2}\pm\sqrt{\left(l+\frac{1}{2}\right)^2}=-\frac{1}{2}\pm\left(l+\frac{1}{2}\right)$$

$$\alpha_1=l, \quad \alpha_2=-l-1$$

$$0\leq x<t \quad G_1(x,t)=C_1x^l \qquad y(0)=0$$

$$t<x<\infty \quad G_2(x,t)=C_3x^l+C_2x^{-l-1} \qquad y(\infty)=0 \rightarrow C_3=0$$

$$=C_2x^{-l-1}$$

Continuity: $C_1t^l-C_2t^{-l-1}=0$

derivative $C_2(-l-1)t^{-l-2}-C_1lt^{l-1}=-\frac{1}{t^2}$

The Wronskian $\begin{vmatrix} x^l & x^{-l-1} \\ lx^{l-1} & (-l-1)x^{-l-2} \end{vmatrix}=(-l-1)x^{-2}-lx^{-2}=-\frac{1}{x^2}$

$$\Rightarrow \begin{cases} C_1=-t^{-l-1} \\ C_2=t^l \end{cases} \Rightarrow \begin{cases} G_1(x,t)=-t^{-l-1}\cdot x^l \\ G_2(x,t)=t^lx^{-l-1} \end{cases}$$

9. A plane wave may be expanded in a series of spherical waves by the Rayleigh equation

$$e^{ikr\cos\gamma} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\gamma)$$

Show that $a_n = i^n(2n+1)$ by using the following steps: (6p)

(a) Use the orthogonality of the P_n functions to solve for $a_n j_n(kr)$

(b) Differentiate n times with respect to kr and set r=0 to eliminate the r-dependence

(c) Evaluate the remaining integral by the relation

$$\int_{-1}^{1} x^n P_n(x)dx = \frac{2^{n+1} n! n!}{(2n+1)!}$$

9. (i) Find $a_n j_n(kr)$. Multiply by $P_{n'}(\cos\gamma)$ and integrate

$$\int_0^\pi e^{ikr\cos\gamma} P_{n'}(\cos\gamma)\sin\gamma\, d\gamma = \sum_{n=0}^{\infty} a_n j_n(kr) \underbrace{\int_0^\pi P_n(\cos\gamma) P_{n'}(\cos\gamma)\sin\gamma\, d\gamma}_{\frac{2\delta_{nn'}}{2n+1}}$$

$$= \frac{2}{2n'+1} a_{n'} j_{n'}(kr)$$

(ii) Differentiate n times with respect to kr

LHS: $\frac{d^n}{d(kr)^n}\int_0^\pi e^{ikr\cos\gamma} P_{n'}(\cos\gamma)\sin\gamma\, d\gamma = \int_0^\pi (i\cos\gamma)^n e^{ikr\cos\gamma} P_{n'}(\cos\gamma)\sin\gamma\, d\gamma =$

$$= i^n \int_0^\pi (\cos\gamma)^n e^{ikr\cos\gamma} P_{n'}(\cos\gamma)\sin\gamma\, d\gamma$$

RHS: $\frac{2a_n}{2n+1}\frac{d^n}{d(kr)^n} j_n(kr)\Big|_{r=0} = \frac{2a_n}{2n+1}\frac{d^n}{d(kr)^n}\left\{2^n (kr)^n \sum_{s=0}^{\infty} \frac{(-1)^s (s+n)!}{s!(2s+2n+1)!}(kr)^{2s}\right\}_{r=0}$

$$= \frac{2a_n}{2n+1}\cdot\frac{2^n n! n!}{(2n+1)!}$$ (only s=0 term survives when $r\to 0$)

LHS: $r\to 0$ $\quad i^n \int_0^\pi (\cos\gamma)^n P_n(\cos\gamma)\sin\gamma\, d\gamma = \left[\begin{array}{l}\cos\gamma = x\\ -\sin\gamma\, d\gamma = dx\end{array}\right]$

$$= -i^n \int_1^{-1} x^n P_n(x)dx = i^n \frac{2^{n+1} n! n!}{(2n+1)!} = [\text{RHS}] = \frac{2a_n}{2n+1}\cdot\frac{2^n n! n!}{(2n+1)!}$$

$$\Rightarrow a_n = i^n(2n+1) \quad \text{QED}$$