Written Examination for Mathematical Methods of Physics
2013.01.03 at 10:30-15:30

Allowed help: "Arfken and Weber" (or "Weber and Arfken"), "Physics Handbook", "Beta: Mathematics Handbook"

In order to get full credit:
1) Used formalisms should be clearly defined
2) <u>All steps in your derivations</u> that are based on references in the above books should be clearly given through reference to the relevant equations or tables.

1. Use calculus of residues to evaluate the integral $\int_0^\infty \frac{x^{p-1}dx}{1+x}$ where $0 < p < 1$. (3p)

2. Express the function $f(x) = 35x^4 + 5x^3 + 3x^2 + 3$ in terms of Legendre polynomials (2p)

3. Use the Frobenius method to first find one (simple) solution to the equation: $x^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} - y = 0$ and then construct the second linearly independent solution based on the first. You do not have to do the final integral but you do need to demonstrate that your solutions solve the equation. (4p)

4. Find the inverse Laplace transform of $F(s) = \frac{s^2}{(s^2+a^2)^2}$ by solving the Bromwich integral. (3p)

5. Water at 100° is flowing through a long pipe of radius a rapidly enough so that we can assume that the temperature is 100° at all points. At t=0, the water is turned off and the surface of the pipe is maintained at 40° from then on (neglect the wall thickness) of the pipe). Find the temperature distribution in the water as function of r and t. Because of the symmetry only a cross section of the pipe needs to be considered. (4p)

6. Use Laplace transforms to solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = t^2e^{-2t}$ with initial conditions $y(0) = y_0$ and the derivative $y'(0) = y_0'$. (4p)

7. An oscillator is subject both to a dissipation and a driving force $f(t)$ where

$$f(t) = \begin{cases} 0 & t < 0 \\ \gamma \exp(-t) & t \geq 0 \end{cases}$$

The equation describing the subsequent motion can be written

$\frac{d^2}{dt^2}X(t) + 2\beta\frac{d}{dt}X(t) + \omega_0^2 X(t) = f(t)$. Use the Fourier transform

$\tilde{g}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} G(t)\exp(i\omega t)dt$ to show that the retarded Green function $G_r(t)$ is given by

$$G_r(t,t') = \begin{cases} 0 & t < t' \\ \omega_1^{-1} \exp(-\beta(t-t'))\sin(\omega_1(t-t')) & t > t' \end{cases}$$ where $\omega_1 = \left(\omega_0^2 - \beta^2\right)^{1/2}$. (Retarded Green function has the cause preceding the effect). (4p)

8. The generating function for the Bessel functions is given by $G(x,t) = exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right)$.

a) Use the product $G(x,t)G(y,t) = G(x+y,t)$ to derive the addition theorem (1p)

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y)$$

b) Use the product $G(x,t)G(-x,t) = 1$ to derive the identity (2p)

$$1 = [J_0(x)]^2 + 2\sum_{n=1}^{\infty}[J_n(x)]^2.$$

(Hint: Use the result from **a)** and known properties of the Bessel functions with integer index.)

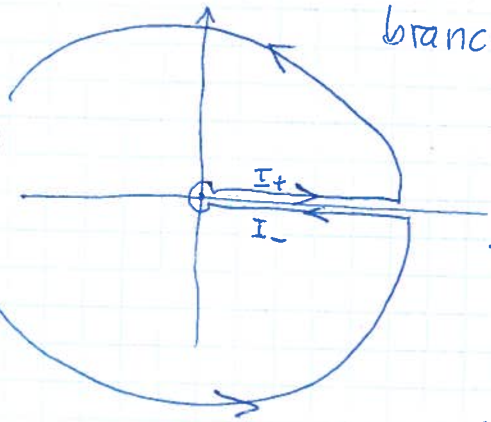
9. Solve the equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 18xe^x$. (3p)

1. Use calculus of residues to evaluate the integral $\int_0^\infty \frac{x^{p-1}dx}{1+x}$ where $0 < p < 1$. (3p)

1

Evaluate: $\int_0^\infty \frac{x^{p-1}dx}{1+x}$, $0<p<1$

Extend into the complex plane and take contour with branch cut to make the integrand single-valued



$$\oint_\Gamma \frac{z^{p-1}dz}{1+z} = 2\pi i \sum \{\text{contained residues}\}$$

There is a pole at $z=-1$ with residue $(e^{i\pi})^{p-1} = -e^{i\pi p}$ $\Rightarrow \oint_\Gamma \frac{z^{p-1}dz}{1+z} = -2\pi i e^{i\pi p}$

Along the two circles $z = re^{i\vartheta}$ giving the contributions

$$\int \frac{r^{p-1}e^{i(p-1)\vartheta}}{1+re^{i\vartheta}} rie^{i\vartheta}d\vartheta = i\int \frac{r^p e^{ip\vartheta}d\vartheta}{1+re^{i\vartheta}} \to 0 \text{ for } r\to 0 \text{ or } r\to\infty$$

Along the branch cut $I_+ = \int_0^\infty \frac{r^{p-1}dr}{1+r}$

Along the branch cut $I_- = \int_\infty^0 \frac{r^{p-1}e^{2\pi i(p-1)}}{1+re^{2\pi i}} e^{2\pi i}dr = -e^{2\pi i p}\int_0^\infty \frac{r^{p-1}dr}{1+r}$

$$\therefore (1-e^{2\pi i p})\int_0^\infty \frac{r^{p-1}}{1+r}dr = -2\pi i e^{i\pi p}$$

$$\int_0^\infty \frac{r^{p-1}dr}{1+r} = \frac{2\pi i e^{i\pi p}}{(e^{2\pi i p}-1)} = \frac{2\pi i e^{i\pi p}}{e^{i\pi p}(e^{i\pi p}-e^{-i\pi p})} = \frac{\pi}{\sin \pi p}$$

$$y'' + y' - 2y = 18\,x\,e^{x}$$

Solve homogeneous equation first:

$$y''+y'-2y=0 \qquad \text{make ansatz} \quad y(x)=e^{\lambda x}$$

$$\Rightarrow (\lambda^2+\lambda-2)y=0 \qquad \text{roots} \quad \lambda^2+\lambda-2=0$$

$$\left(\lambda+\tfrac{1}{2}\right)^2-\tfrac{1}{4}=2$$

$$\lambda=-\tfrac{1}{2}\pm\tfrac{3}{2} \Rightarrow \begin{cases}\lambda_1=1\\ \lambda_2=-2\end{cases}$$

Homogeneous solution $y(x)=A\,e^{x}+B\,e^{-2x}$

Particular solution from

(A&W 9.6.25) $$y_p(x)=y_2(x)\int^x \frac{y_1(s)F(s)}{W\{y_1(s),y_2(s)\}}\,ds-y_1(x)\int^x \frac{y_2(s)F(s)}{W\{y_1(s),y_2(s)\}}\,ds$$

where $y_1(x)=e^{x}$, $y_2(x)=e^{-2x}$ and $F(x)=18\,x\,e^{x}$

~~$y_p(x)=e^2$~~ $$W\{y_1,y_2\}=\begin{vmatrix} e^{x} & e^{-2x}\\ e^{x} & -2e^{-2x}\end{vmatrix}=-3e^{-x}$$

$$y_p(x)=e^{-2x}\int^x \frac{e^{s}\cdot 18se^{s}}{(-3e^{-s})}\,ds-e^{x}\int^x \frac{e^{-2s}\,18se^{s}}{(-3e^{-s})}\,ds=$$

$$=-\cancel{18}\,6\,e^{-2x}\int^x s\,e^{3s}\,ds+\cancel{18}\,6\,e^{x}\int^x s\,ds=$$

$$=-6\,e^{-2x}\left\{\left[s\frac{e^{3s}}{3}\right]^x-\frac{1}{3}\int^x e^{3s}\,ds\right\}+6\,e^{x}\left[\tfrac{1}{2}s^2\right]^x=$$

$$=-2x\,e^{x}+2\,e^{-2x}\left[\frac{e^{3s}}{3}\right]^x+3x^2e^{x}=$$

$$=(3x^2-2x)e^{x}+2\cdot y_1 \Rightarrow y_p(x)=(3x^2-2x)e^{x}$$

2. Express the function $f(x) = 35x^4 + 5x^3 + 3x^2 + 3$ in terms of Legendre polynomials

(2p)

2. $f(x) = 35x^4 + 5x^3 + 3x^2 + 3$

The highest degree is 4 -> P_4

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \iff 35x^4 = 8P_4(x) + 30x^2 - 3$

$\therefore f(x) = 8P_4(x) + 5x^3 + 33x^2$

Highest remaining power is 3 $\rightarrow P_3$

$P_3(x) = \frac{1}{2}(5x^3 - 3x) \iff 5x^3 = 2P_3(x) + 3x$

$\therefore f(x) = 8P_4(x) + 2P_3(x) + 33x^2 + 3x$

Highest remaining power is 2 $\rightarrow P_2$

$P_2(x) = \frac{1}{2}(3x^2 - 1) \iff 3x^2 = 2P_2(x) + 1$

$\therefore f(x) = 8P_4(x) + 2P_3(x) + 22P_2(x) + 3x + 11$

$P_1(x) = x, \quad P_0(x) = 1$

$\therefore f(x) = 8P_4(x) + 2P_3(x) + 22P_2(x) + 3P_1(x) + 11P_0(x)$

$$= \frac{1}{3}t^3e^{-2t} - \frac{1}{6}t^4e^{-2t} - 4y_0te^{-2t} + y_0'e^{-2t} - 2y_0'te^{-2t}$$

$$y'' = \underline{t^2e^{-2t}} - \underline{\frac{2}{3}t^3e^{-2t}} - \underline{\frac{2}{3}t^3e^{-2t}} + \underline{\frac{1}{3}t^4e^{-2t}} - \underline{4y_0e^{-2t}} + \underline{8y_0te^{-2t}} -$$

$$- \underline{2y_0'e^{-2t}} - \underline{2y_0'e^{-2t}} + 4y_0'te^{-2t} =$$

$$= -\frac{4}{3}\underline{t^3e^{-2t}} + \frac{1}{3}\underline{t^4e^{-2t}} + \underline{t^2e^{-2t}} - 4\underline{y_0'}e^{-2t} - 4\underline{y_0}e^{-2t} + 8\underline{y_0}te^{-2t}$$

$$+ 4y_0'te^{-2t}$$

$$y'' + 4y' + 4y = \cancel{\frac{1}{3}t^4} - \cancel{\frac{4}{3}t^3} + t^2 - \cancel{4y_0'} - \cancel{4y_0} + \cancel{8y_0}t + \cancel{4y_0'}t +$$

$$+ \cancel{\frac{4}{3}t^3} - \cancel{\frac{2}{3}t^4} - \cancel{16y_0}t + \cancel{4y_0'} - \cancel{8y_0'}t +$$

$$+ \cancel{\frac{1}{3}t^4} + \cancel{4y_0} - \cancel{8t}y_0 + \cancel{16y_0}t + \cancel{4y_0'}t = t^2$$

$$y(0) = y_0$$
$$y'(0) = y_0'$$

3. Use the Frobenius method to first find one (simple) solution to the equation: $x^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}-y=0$ and then construct the second linearly independent solution based on the first. You do not have to do the final integral but you do need to demonstrate that your solutions solve the equation. (4p)

The ansatz $y(x)=\sum_k a_k x^{k+\lambda}$ gives

$$a_k\{(\lambda+k)(\lambda+k-1)+\lambda+k-1\}+a_{k+1}(\lambda+k+1)=0$$

With indicial equation $\lambda a_0 x^{\lambda-1}=0 \rightarrow \lambda=0$

$$a_{k+1}=-\left\{\frac{k(k-1)+k-1}{k+1}\right\}a_k=-\frac{(k+1)(k-1)}{k+1}a_k=(1-k)a_k$$

$a_1=a_0$

$a_2=0=a_j \quad j>2$, series terminates

$\therefore\ y_1(x)=a_0(1+x)$

Make the ansatz $y_2(x)=u(x)y_1(x)$

$$y_2'=u'y_1+uy_1', \quad y_2''=u''y_1+2u'y_1'+uy_1''$$

The equation becomes: $x^2u''y_1+2x^2u'y_1'+x^2uy_1''+(x+1)(u'y_1+uy_1')-$

$-uy_1=u\{\underbrace{x^2y_1''+(x+1)y_1'-uy_1}_{=0}\}+x^2u''y_1+2x^2u'y_1'+(x+1)u$

$=0$

$$\Rightarrow x^2\frac{u''}{u'}y_1+2x^2y_1'+(x+1)y_1=0$$

$$\Leftrightarrow \frac{u''}{u'}=-\frac{2y_1'}{y_1}-\frac{x+1}{x^2} \Rightarrow \frac{du'}{u'}=-\left\{\frac{2}{1+x}+\frac{1}{x}+\frac{1}{x^2}\right\}dx$$

$$\ln u'=-2\ln(1+x)-\ln x+\frac{1}{x}+\ln C$$

$$u'=\frac{C}{x(1+x)^2}e^{1/x}$$

$$u(x)=C\int^x\frac{e^{1/s}}{s(1+s)^2}ds=C\frac{xe^{1/x}}{1+x}$$

$$\Rightarrow y_2(x)=Cxe^{1/x} \quad \text{or} \quad y_2(x)=(1+x)\int^x\frac{e^{1/s}}{s(1+s)^2}ds$$

3 cont'd

a) Without doing the integral:

$$y_2(x) = (1+x)\int^x \frac{e^{1/s}ds}{s(1+s)^2} \quad ; \quad y_2' = \int^x \frac{e^{1/s}ds}{s(1+s)^2} + (1+x)\frac{e^{1/x}}{x(1+x)^2} =$$

$$= \int^x \frac{e^{1/s}ds}{s(1+s)^2} + \frac{e^{1/x}}{x(1+x)}$$

$$y_2'' = \frac{e^{1/x}}{x(1+x)^2} - \frac{e^{1/x}}{x^3(1+x)} - \frac{e^{1/x}}{x^2(1+x)} - \frac{e^{1/x}}{x(1+x)^2} = -\frac{e^{1/x}}{x^3(1+x)} - \frac{e^{1/x}}{x^2(1+x)}$$

The equation:
$$x^2y'' + (x+1)y' - y = -\frac{e^{1/x}}{x(1+x)} - \frac{e^{1/x}}{1+x} + (x+1)\int^x \frac{e^{1/s}ds}{s(1+s)^2} + \frac{e^{1/x}}{x} -$$

$$- (1+x)\int^x \frac{e^{1/s}ds}{s(1+s)^2} = \frac{e^{1/x}}{x(1+x)}\left\{-1 - x + 1 + x\right\} = 0$$

b) From $y_2(x) = xe^{1/x}$; $y_2' = e^{1/x} - \frac{e^{1/x}}{x}$; $y_2'' = -\frac{1}{x^2}e^{1/x} + \frac{1}{x^2}e^{1/x} + \frac{e^{1/x}}{x^3}$

$$= \frac{e^{1/x}}{x^3}$$

The equation:
$$x^2y'' + (x+1)y' - y = \frac{e^{1/x}}{x} + (x+1)\left(e^{1/x} - \frac{e^{1/x}}{x}\right) - xe^{1/x} =$$

$$= \cancel{\frac{e^{1/x}}{x}} + \cancel{xe^{1/x}} - \cancel{e^{1/x}} + \cancel{e^{1/x}} - \cancel{\frac{e^{1/x}}{x}} - \cancel{xe^{1/x}} = 0$$

4. Find the inverse Laplace transform of $F(s)=\dfrac{s^2}{\left(s^2+a^2\right)^2}$ by solving the Bromwich integral.

(3p)

$$F(s)=\frac{s^2}{(s^2+a^2)^2} \longrightarrow F(t)=\frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma}\frac{s^2e^{st}}{(s^2+a^2)^2}\,ds$$

Double poles at $s=\pm ia$

Residue at $s=+ia$: $\displaystyle \frac{d}{ds}\frac{\cancel{(s-ia)^2}\,s^2e^{st}}{\cancel{(s-ia)^2}(s+ia)^2}\Big|_{s=ia}=$

$$=\frac{2se^{st}}{(s+ia)^2}+\frac{ts^2e^{st}}{(s+ia)^2}-2\frac{s^2e^{st}}{(s+ia)^3}\Big|_{s=ia}=$$

$$=\frac{2ia\,e^{iat}}{(2ia)^2}+\frac{t(ia)^2e^{iat}}{(2ia)^2}-\frac{2(ia)^2e^{iat}}{(2ia)^3}=$$

$$=\frac{e^{iat}}{2ia}+\frac{1}{4}te^{iat}-\frac{1}{2a}\cdot\frac{e^{iat}}{2i}$$

Residue at $s=-ia$: $\displaystyle \frac{d}{ds}\frac{\cancel{(s+ia)^2}\,s^2e^{st}}{(s-ia)^2\cancel{(s+ia)^2}}\Big|_{s=-ia}=$

$$=\frac{2se^{st}}{(s-ia)^2}+\frac{ts^2e^{st}}{(s-ia)^2}-\frac{2s^2e^{st}}{(s-ia)^3}\Big|_{s=-ia}=$$

$$=-\frac{2ia\,e^{-iat}}{(-2ia)^2}+\frac{t(-ia)^2e^{-iat}}{(-2ia)^2}-\frac{2(-ia)^2e^{-iat}}{(-2ia)^3}=$$

$$=\frac{-1}{2ia}\cdot e^{-iat}+\frac{1}{4}te^{-iat}+\frac{1}{2a}\cdot\frac{e^{-iat}}{2i}$$

$$\therefore F(t)=\frac{1}{a}\Big\{\underbrace{\frac{e^{iat}-e^{-iat}}{2i}}_{\sin(at)}\Big\}+\frac{1}{2}t\Big\{\underbrace{\frac{e^{iat}+e^{-iat}}{2}}_{\cos(at)}\Big\}-\frac{1}{2a}\Big\{\underbrace{\frac{e^{iat}-e^{-iat}}{2i}}_{\sin(at)}\Big\}=$$

$$=\frac{1}{2a}\sin(at)+\frac{1}{2}t\cos(at)$$

5. Water at 100° is flowing through a long pipe of radius a rapidly enough so that we can assume that the temperature is 100° at all points. At t=0, the water is turned off and the surface of the pipe is maintained at 40° from then on (neglect the wall thickness) of the pipe). Find the temperature distribution in the water as function of r and t. Because of the symmetry only a cross section of the pipe needs to be considered. (4p)

Use cylindrical coordinates without z-dependence

$$\frac{\partial u}{\partial t} = \varkappa^2 \left\{ \frac{1}{\varrho}\frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial u}{\partial \varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2 u}{\partial \varphi^2} \right\}$$

Take $u(\varrho,\varphi,t) = 40 + v(\varrho,\varphi,t)$ to have homogeneous boundary conditions. Make the ansatz $v(\varrho,\varphi,t) = R(\varrho)\,\Phi(\varphi)\,T(t)$

Separate the time-dependence

$$\frac{1}{\varkappa^2 T}\frac{dT}{dt} = \frac{1}{\varrho R}\frac{d}{d\varrho}\left(\varrho \frac{dR}{d\varrho}\right) + \frac{1}{\varrho^2 \Phi}\frac{d^2\Phi}{d\varphi^2} = -\lambda^2$$

$$\Rightarrow \frac{dT}{dt} + (\varkappa\lambda)^2 T = 0 \quad \to \quad T(t) = e^{-(\varkappa\lambda)^2 t}$$

$$\frac{1}{\varrho R}\frac{d}{d\varrho}\left(\varrho \frac{dR}{d\varrho}\right) + \frac{1}{\varrho^2 \Phi}\frac{d^2\Phi}{d\varphi^2} = -\lambda^2$$

scale by ϱ^2 to separate

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -\frac{\varrho}{R}\frac{d}{d\varrho}\left(\varrho \frac{dR}{d\varrho}\right) - \lambda^2\varrho^2 = -n^2$$

$$\Rightarrow \frac{d^2\Phi}{d\varphi^2} + n^2\Phi = 0 \quad \to \quad \Phi(\varphi) = A\cos n\varphi + B\sin n\varphi$$

The symmetry gives $n=0$ and $\Phi(\varphi) = 1$

$$\varrho\frac{d}{d\varrho}\left(\varrho \frac{dR}{d\varrho}\right) + \lambda^2\varrho^2 R = 0 \qquad \text{Bessel equation for } n=0$$

Since the origin is included we have to exclude Neumann function solutions, thus:

$$v(\varrho,\varphi,t) \equiv v(\varrho,t) = J_0(\lambda\varrho)\,e^{-(\varkappa\lambda)^2 t}$$

5, cont'd Boundary condition $v(a,t)=0$

$$\Rightarrow v(\rho,t) \longrightarrow v_n(\rho,t) = J_0\left(\alpha_n \frac{\rho}{a}\right) e^{-\left(\frac{\kappa \alpha_n}{a}\right)^2 t}$$

with α_n a zero of J_0

Initially: $u(\rho,0)=100 \longrightarrow v(\rho,0)=60$

Set up a general solution

$$v(\rho,t) = \sum_{n=1}^{\infty} c_n J_0\left(\alpha_n \frac{\rho}{a}\right) e^{-\left(\frac{\kappa \alpha_n}{a}\right)^2 t}$$

and $v(\rho,0) = \sum_{n=1}^{\infty} c_n J_0\left(\alpha_n \frac{\rho}{a}\right) = 60$

Project to find the coefficients c_n:

$$60 \int_0^a J_0\left(\alpha_m \frac{\rho}{a}\right) \rho \, d\rho = \sum_{n=1}^{\infty} c_n \int_0^a J_0\left(\alpha_m \frac{\rho}{a}\right) J_0\left(\alpha_n \frac{\rho}{a}\right) \rho \, d\rho =$$

$$= \sum_{n=1}^{\infty} c_n \delta_{nm} \frac{a^2}{2} \left[J_1(\alpha_m)\right]^2 = \frac{c_m a^2}{2} \left[J_1(\alpha_m)\right]^2$$

$$\int_0^a J_0\left(\alpha_m \frac{\rho}{a}\right) \rho \, d\rho = \left[\begin{array}{l} \frac{\alpha_m \rho}{a} = x \\ \rho = \frac{a}{\alpha_m} x \\ \rho = a \rightarrow x = \alpha_m \end{array}\right] = \left(\frac{a}{\alpha_m}\right)^2 \int_0^{\alpha_m} J_0(x) x \, dx =$$

$$= \left(\frac{a}{\alpha_m}\right)^2 \int_0^{\alpha_m} \frac{d}{dx}\left\{x J_1(x)\right\} dx = \left(\frac{a}{\alpha_m}\right)^2 \left[x J_1(x)\right]_0^{\alpha_m} =$$

$$= \frac{a^2}{\alpha_m} J_1(\alpha_m)$$

$$\therefore \quad c_m = \frac{120}{a^2 [J_1(\alpha_m)]^2} \cdot \frac{a^2}{\alpha_m} J_1(\alpha_m) = \frac{120}{\alpha_m J_1(\alpha_m)}$$

$$\Rightarrow u(\rho,t) = 40 + 120 \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\alpha_n \rho}{a}\right)}{\alpha_n J_1(\alpha_n)} e^{-\left(\frac{\kappa \alpha_n}{a}\right)^2 t}$$

6. Use Laplace transforms to solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = t^2e^{-2t}$ with initial conditions $y(0) = y_0$ and the derivative $y'(0) = y_0'$. (4p)

The transformed equation is

$$s^2\tilde{y} - sy_0 - y_0' + 4s\tilde{y} - 4y_0 + 4\tilde{y} = \frac{2}{(s+2)^3}$$

$$\Rightarrow (s^2+4s+4)\tilde{y} = \frac{2}{(s+2)^3} + (s+4)y_0 + y_0'$$

$$\tilde{y} = \frac{2}{(s+2)^5} + \frac{s+4}{(s+2)^2}y_0 + \frac{y_0'}{(s+2)^2}$$

Take the inverse transform of each term:

1) $y_1(t) = \frac{2}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} \frac{e^{st}ds}{(s+2)^5}$; 5^{th} order pole at $s=-2$

residue $\frac{1}{4!}\frac{d^{(IV)}}{ds^{(IV)}}\frac{(s+2)^5e^{st}}{(s+2)^5}\Big|_{s=-2} = \frac{t^4e^{-2t}}{24}$

$$\Rightarrow y_1(t) = \frac{1}{12}t^4e^{-2t}$$

2) $y_2(t) = \frac{y_0}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma}\frac{se^{st}ds}{(s+2)^2}$; 2^{nd} order pole at $s=-2$

residue: $\frac{d}{ds}\frac{(s+2)^2se^{st}}{(s+2)^2}\Big|_{s=-2} = e^{-2t} + t(-2)e^{-2t}$;

$$y_2(t) = y_0(e^{-2t} - 2te^{-2t})$$

3) $y_3(t) = \frac{4y_0+y_0'}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma}\frac{e^{st}}{(s+2)^2}ds$; 2^{nd} order pole at $s=-2$

residue: $\frac{d}{ds}\frac{(s+2)^2e^{st}}{(s+2)^2}\Big|_{s=-2} = te^{-2t}$

$$y_3(t) = (4y_0+y_0')te^{-2t}$$

$$\therefore\ y(t) = \frac{1}{12}t^4e^{-2t} + y_0(e^{-2t} - 2te^{-2t}) + (4y_0+y_0')te^{-2t} =$$

$$= \frac{1}{12}t^4e^{-2t} + y_0e^{-2t} + (2y_0+y_0')te^{-2t}$$

6 cont'd

Check the solution:

$$y' = \left\{ \frac{1}{3}t^3 + \cancel{2y_0} + y_0' - \frac{1}{6}t^4 - \cancel{2y_0} - 4y_0 t - 2y_0' t \right\} e^{-2t} =$$

$$= \left\{ \frac{1}{3}t^3 - \frac{1}{6}t^4 - 4y_0 t - 2y_0' t + y_0' \right\} e^{-2t}$$

$$y'' = \left\{ t^2 - \frac{2}{3}t^3 - 4y_0 - 2y_0' - \frac{2}{3}t^3 + \frac{1}{3}t^4 + 8y_0 t + 4y_0' t - 2y_0' \right\} e^{-2t} =$$

$$= \left\{ \frac{1}{3}t^4 - \frac{4}{3}t^3 + t^2 + 8y_0 t + 4y_0' t - 4y_0 - 4y_0' \right\} e^{-2t}$$

$$y'' + 4y' + 4y = \left\{ \cancel{\frac{1}{3}t^4} - \cancel{\frac{4}{3}t^3} + t^2 + \cancel{8y_0 t} + \cancel{4y_0' t} - \cancel{4y_0} - \cancel{4y_0'} + \right.$$

$$\left. + \cancel{\frac{4}{3}t^3} - \cancel{\frac{2}{3}t^4} - \cancel{16y_0 t} - \cancel{8y_0' t} + \cancel{4y_0'} + \cancel{\frac{1}{3}t^4} + \cancel{4y_0} + \cancel{8y_0 t} + \cancel{4y_0' t} \right\} e^{-2t}$$

$$= t^2 e^{-2t} \quad \text{ok}$$

$y(0) = y_0$ ok

$y'(0) = y_0'$ ok

x ——— Alternative ——— x

Using convolution to get $y_1(t)$

$$\tilde{y}_1(s) = \frac{\mathcal{L}\{t^2 e^{-2t}\}}{(s+2)^2} \quad ; \quad \mathcal{L}^{-1}\left\{ \frac{1}{(s+2)^2} \right\} = t e^{-2t} \quad (\text{from } 3)$$

$$y_1(t) = \int_0^t (t-z) e^{-2(t-z)} z^2 e^{-2z} dz = \int_0^t t e^{-2(t-z)} z^2 e^{-2z} dz - \int_0^t z^3 e^{-2(t-z)} e^{-2z} dz =$$

$$= t e^{-2t} \left[\frac{z^3}{3} \right]_0^t - e^{-2t} \left[\frac{z^4}{4} \right]_0^t = \frac{1}{3} t^4 e^{-2t} - \frac{1}{4} t^4 e^{-2t} = \frac{1}{12} t^4 e^{-2t}$$

7. An oscillator is subject both to a dissipation and a driving force $f(t)$ where

$$f(t)=\begin{cases}0 & t<0\\ \gamma\exp(-t) & t\geq 0\end{cases}$$

The equation describing the subsequent motion can be written $\frac{d^2}{dt^2}X(t)+2\beta\frac{d}{dt}X(t)+\omega_0^2X(t)=f(t)$. Use the Fourier transform $\tilde{g}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}G(t)\exp(i\omega t)dt$ to show that the retarded Green function $G_r(t)$ is given by

$$G_r(t,t')=\begin{cases}0 & t<t'\\ \omega_1^{-1}\exp(-\beta(t-t'))\sin(\omega_1(t-t')) & t>t'\end{cases}$$ where $\omega_1=(\omega_0^2-\beta^2)^{1/2}$. (Retarded Green function has the cause preceding the effect).

(4p)

The equation for the Green function

$$\ddot{G}+2\beta\dot{G}+\omega_0^2G=\delta(t'-t) \quad \text{(note } +\delta(t'-t) \text{ since RHS is } +f(t)\text{)}$$

Fourier transform $G\to\tilde{g}$, $\mathcal{L}\{\dot{G}\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dG}{dt}e^{i\omega t}dt=$

$$=\frac{1}{\sqrt{2\pi}}\left\{\left[Ge^{i\omega t}\right]_{-\infty}^{\infty}-i\omega\int_{-\infty}^{\infty}Ge^{i\omega t}dt\right\}=(-i\omega)\tilde{g} \quad (\text{Assuming } G\to 0 \text{ for } t\to\pm\infty)$$

$$\mathcal{L}\{\ddot{G}\}=(-i\omega)^2\tilde{g}\ ;\ \mathcal{L}\{\delta(t'-t)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t'-t)e^{i\omega t}dt=\frac{1}{\sqrt{2\pi}}e^{i\omega t'}$$

Transformed equation $(-\omega^2-2i\beta\omega+\omega_0^2)\tilde{g}=\frac{e^{i\omega t'}}{\sqrt{2\pi}}$

$$\Rightarrow \tilde{g}(\omega)=\frac{1}{\sqrt{2\pi}}\cdot\frac{e^{i\omega t'}}{\omega_0^2-\omega^2-2i\beta\omega}$$

Back transform $G(t,t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega(t-t')}}{\omega_0^2-\omega^2-2i\beta\omega}d\omega$

We have simple poles at $\omega^2+2i\beta\omega-\omega_0^2=0$

$$(\omega+i\beta)^2=\omega_0^2-\beta^2\equiv\omega_1^2$$

$\omega=-i\beta\pm\omega_1$ Poles in the lower half plane

For $t<t'$ we close in the upper half plane, the integrand vanishes on the half circle as $R\to\infty$. No poles are contained so $G(t,t')=0 \quad t<t'$

7, cont'd For $t > t'$ we close in the lower half plane. The contour is traversed clockwise and again the half circle does not contribute:

$$G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t')}}{\omega_0^2-\omega^2-2i\beta\omega}\,d\omega = -i\,\Sigma\{\text{enclosed residues}$$

Write the denominator as a product

$$\omega_0^2-\omega^2-2i\beta\omega = -\{\omega^2+2i\beta\omega-\omega_0^2\} = -(\omega+i\beta+\omega_1)(\omega+i\beta-\omega_1)$$

The residues at $z = -i\beta \pm \omega_1$

$$\lim_{z\to -i\beta+\omega_1} -\frac{\cancel{(z+i\beta-\omega_1)}\,e^{-iz(t-t')}}{\cancel{(z+i\beta-\omega_1)}(z+i\beta+\omega_1)} = -\frac{e^{-i(-i\beta+\omega_1)(t-t')}}{2\omega_1} =$$

$$= -\frac{e^{-\beta(t-t')}}{2\omega_1}\,e^{-i\omega_1(t-t')}$$

$$\lim_{z\to -i\beta-\omega_1} -\frac{\cancel{(z+i\beta+\omega_1)}\,e^{-iz(t-t')}}{(z+i\beta-\omega_1)\cancel{(z+i\beta+\omega_1)}} = -\frac{e^{-i(-i\beta-\omega_1)(t-t')}}{-2\omega_1} =$$

$$= \frac{e^{-\beta(t-t')}}{2\omega_1}\,e^{i\omega_1(t-t')}$$

$$\therefore \begin{cases} G(t,t') = -i\dfrac{e^{-\beta(t-t')}}{2\omega_1}\left\{e^{i\omega_1(t-t')} - e^{-i\omega_1(t-t')}\right\} = \\ \qquad = \dfrac{e^{-\beta(t-t')}}{\omega_1}\sin\omega_1(t-t') \quad \text{for } t > t' \\ G(t,t') = 0 \quad \text{for } t < t' \end{cases}$$

8. The generating function for the Bessel functions is given by $G(x,t) = exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right)$.

a) Use the product $G(x,t)G(y,t) = G(x+y,t)$ to derive the addition theorem (1p)

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y)$$

b) Use the product $G(x,t)G(-x,t) = 1$ to derive the identity (2p)

$$1 = [J_0(x)]^2 + 2\sum_{n=1}^{\infty} [J_n(x)]^2.$$

(Hint: Use the result from **a)** and known properties of the Bessel functions with integer index.)

8 a

$$G(x+y,t) = \sum_{k=-\infty}^{\infty} J_k(x+y) t^k = \sum_{n=-\infty}^{\infty} J_n(x) t^n \sum_{m=-\infty}^{\infty} J_m(y) t^m =$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} t^{m+n} J_n(x) J_m(y) \; ; \; \text{put } k = m+n$$

$$\Rightarrow \quad = \sum_{k=-\infty}^{\infty} t^k \sum_{n=-\infty}^{\infty} J_n(x) J_{k-n}(y)$$

$$\therefore \; J_k(x+y) = \sum_{n=-\infty}^{\infty} J_n(x) J_{k-n}(y)$$

b)

$$1 = \sum_{k=-\infty}^{\infty} t^k J_k(0) = \sum_{k=-\infty}^{\infty} t^k \sum_{n=-\infty}^{\infty} J_n(x) J_{k-n}(-x)$$

but $J_k(0) = 0$, $k \neq 0 \Rightarrow$

$$1 = J_0(0) = \sum_{n=-\infty}^{\infty} J_n(x) J_{-n}(-x) = [J_0(x)]^2 + \sum_{m=1}^{\infty} \{ J_m(x) J_{-m}(-x) + J_{-m}(x) J_m(-x) \}$$

$$= [J_0(x)]^2 + 2\sum_{m=1}^{\infty} [J_m(x)]^2$$

since $J_{-m}(x) = (-1)^m J_m(x)$

and $J_m(-x) = (-1)^m J_m(x)$

$$y_2(x) = (1+x)\int^{x} \frac{e^{1/s}}{s(1+s)^2}\,ds$$

$$y_2' = \int^{x} \frac{e^{1/s}}{s(1+s)^2}\,ds + \cancel{(1+x)}\,\frac{e^{1/x}}{x(1+x)^{\cancel{2}}}$$

$$y_2'' = 2\,\frac{e^{1/x}}{x(1+x)^2} - (1+x)\left\{ \frac{e^{1/x}}{x^3(1+x)^2} + \frac{e^{1/x}}{x^2(1+x)^2} + \frac{2e^{1/x}}{x(1+x)^3} \right\} =$$

$$= 2\,\frac{e^{1/x}}{x(1+x)^2} - \frac{e^{1/x}}{x^3(1+x)} - \frac{e^{1/x}}{x^2(1+x)} - \frac{2e^{1/x}}{x(1+x)^2}$$

$$x^2 y_2'' + (x+1)y_2' - y_2 = \cancel{\frac{2x e^{1/x}}{(1+x)^2}} - \frac{e^{1/x}}{x(1+x)} - \frac{e^{1/x}}{1+x} - \cancel{\frac{2x e^{1/x}}{(1+x)^2}} +$$

$$+ \cancel{(1+x)\int^{x} \frac{e^{1/s}}{s(1+s)^2}\,ds} + \frac{e^{1/x}}{x} - \cancel{(1+x)\int^{x} \frac{e^{1/s}}{s(1+s)^2}\,ds}$$

$$= e^{1/x}\left\{ \frac{1}{x} - \frac{1}{x(1+x)} - \frac{1}{1+x} \right\} = e^{1/x}\left\{ \frac{1+x-1-x}{x(x+1)} \right\} = 0$$

$$\frac{d}{dx}\left(\frac{x e^{1/x}}{1+x}\right) = \frac{e^{1/x}}{1+x} - \frac{e^{1/x}}{x(1+x)} - \frac{x e^{1/x}}{(1+x)^2} =$$

$$= \left\{ \frac{x(1+x) - (1+x) - x^2}{x(1+x)^2} \right\} e^{1/x} =$$

$$= \left\{ \frac{\cancel{x} + \cancel{x^2} - 1 - \cancel{x} - \cancel{x^2}}{x(1+x)^2} \right\} e^{1/x} = -\frac{e^{1/x}}{x(1+x)^2}$$

9. Solve the equation $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-2y=18xe^x$. (3p)

9. $y''+y'-2y=18xe^x$

Solve the homogeneous equation first:

$y''+y'-2y=0$; make the ansatz $y(x)=e^{\lambda x}$

$(\lambda^2+\lambda-2)e^{\lambda x}=0 \Rightarrow$ roots $\lambda^2+\lambda-2=0 \Rightarrow \lambda=\begin{cases}1\\-2\end{cases}$

Homogeneous solution $y(x)=Ae^x+Be^{-2x}$

Particular solution from (A&W 9.6.25):

$$y_p(x)=y_2(x)\int^x\frac{y_1(s)F(s)}{W\{y_1(s),y_2(s)\}}ds-y_1(x)\int^x\frac{y_2(s)F(s)}{W\{y_1(s),y_2(s)\}}ds$$

where $y_1(x)=e^x$, $y_2(x)=e^{-2x}$ and $F(x)=18xe^x$

The Wronskian $W(x)=\begin{vmatrix}e^x & e^{-2x}\\ e^x & -2e^{-2x}\end{vmatrix}=-3e^{-x}$

$$y_p(x)=e^{-2x}\int^x\frac{e^s18se^s}{(-3e^{-s})}ds-e^x\int^x\frac{e^{-2s}18se^s}{(-3e^{-s})}ds=$$

$$=-6e^{-2x}\int^x se^{3s}ds+6e^x\int^x s\,ds=-6e^{-2x}\left\{\left[\tfrac{1}{3}se^{3s}\right]-\tfrac{1}{3}\int e^{3s}ds\right\}+$$

$$+6e^x\left[\tfrac{1}{2}s^2\right]^x=-2e^{-2x}\cdot xe^{3x}+2e^{-2x}\left[\tfrac{1}{3}e^{3s}\right]^x+3x^2e^x=$$

$$=(3x^2-2x)e^x+\tfrac{2}{3}e^x=(3x^2-2x)e^x+\tfrac{2}{3}y_1$$

$\therefore\ y(x)=Ae^x+Be^{-2x}+(3x^2-2x)e^x$

Check: $y'=Ae^x-2Be^{-2x}+(3x^2-2x+6x-2)e^x$

$y''=Ae^x+4Be^{-2x}+(6x\underbrace{-2+6}_{+4}+3x^2+4x-2)e^x=$

$=Ae^x+4Be^{-2x}+(3x^2+10x+2)e^x$

$y''+y'-2y=\cancel{Ae^x}+\cancel{4Be^{-2x}}+(\cancel{3x^2}+10x+\cancel{2})e^x+\cancel{Ae^x}-\cancel{2Be^{-2x}}+(\cancel{3x^2}+\cancel{4}x-\cancel{2})e^x-$

$-\cancel{2Ae^x}-2Be^{-2x}-(\cancel{6x^2}-4x)e^x=18xe^x$

$$F(s) = \frac{s^2}{(s^2+a^2)^2} \longrightarrow F(t) = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} \frac{s^2 e^{st}\,ds}{(s^2+a^2)^2}$$

Double poles at $s = \pm ia$

Residue at $s = +ia$: $\displaystyle \frac{d}{ds}\frac{\cancel{(s-ia)^2}\, s^2 e^{st}}{\cancel{(s-ia)^2}(s+ia)^2}\Bigg|_{s=ia} = \frac{(ia)^2 e^{iat}}{(2ia)^2} =$

$=$