Written Examination for Mathematical Methods of Physics
2012.11.10 at 09:00-14:00

Allowed help: "Arfken and Weber" (or "Weber and Arfken"), "Physics Handbook", "Beta: Mathematics Handbook"

In order to get full credit:
1) Used formalisms should be clearly defined
2) All steps in your derivations that are based on references in the above books should be clearly given through reference to the relevant equations or tables.

Note that problems 8 and 9 are on the back.

1. Use calculus of residues to evaluate the integral $\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx$. Specify which contour you use! (2p)

2. Use the Cauchy integral formula to construct a function $f(z)$ satisfying the properties:
   (a) $f(z)$ is analytic except for a simple pole of residue R at $z=a$ and a branch cut $(0,\infty)$ at which the function has a discontinuity $f(x+i\varepsilon)-f(x-i\varepsilon)=2i\pi g(x)$, $x\geq 0$.
   (b) $|f(z)|\to 0$ as $|z|\to\infty$, and $|zf(z)|\to 0$ as $|z|\to 0$.
   Be careful with the specification of contours used and express the function in terms of R, a, and $g(x)$. (3p)

3. Use the Frobenius method to find the two independent solutions to the equation (one sums to a simple function, the other may be given through the first four terms in the series):
$$3x\frac{d^2y}{dx^2}+(3x+1)\frac{dy}{dx}+y=0 \qquad (4p)$$

4. Find the Fourier transform of $f(x)=\begin{cases}0, & x<0\\ e^{-ax}, & x>0\end{cases}$ where a is real and $a>0$ (3p)

5. Use Laplace transforms to solve $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+13y=20e^{-t}$, $y(0)=1, y'(0)=3$. (4p)

6. Starting from the recursion relation $(2l+1)xP_l = lP_{l-1}+(l+1)P_{l+1}$ for the Legendre polynomials derive the recursion relation for the associated Legendre polynomials:
$(l+1-m)P_{l+1}^m = (2l+1)xP_l^m-(l+m)P_{l-1}^m$ (2p)

7. Solve $(1+x^2)\frac{dy}{dx}+6xy=2x$, $x\in R$ (3p)

8. Use the Laplace transform technique and the defining equation of the Green's function to find the Green's function for the equation $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=f(t)$, where $y(0)=y'(0)=0$ and $y(t)=0$ for $t<0$. (4p)

9. A sphere initially at 0° has its surface kept at 100° from $t=0$ on (for example a frozen pea in boiling water). Find the time-dependent temperature distribution. (Hint: Subtract 100° from all temperatures and solve the problem; then add the 100° to the answer). (5p)

1. Use calculus of residues to evaluate the integral $\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$. Specify which contour you use! (2p)

1. The integral $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} \equiv I$

<u>Solution 1:</u> On the negative real axis the function $f(-x) = i f(x)$ is in one of its two branches and $\displaystyle\int_{-\infty}^{0} \frac{\sqrt{x}\,dx}{1+x^2} = i \int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2}$

Thus, $\displaystyle I = \frac{1}{1+i}\int_{-\infty}^{\infty} \frac{\sqrt{x}\,dx}{1+x^2} = \frac{1}{1+i}\oint \frac{\sqrt{z}\,dz}{1+z^2} = \frac{2\pi i}{1+i}\sum\{\text{residues}\}$

where we close in the upper half plane. The semicircle does not contribute since $\displaystyle\left|\frac{z\cdot\sqrt{z}}{1+z^2}\right| \sim \frac{1}{|z|^{1/2}} \xrightarrow[R\to\infty]{} 0$

The residue at $z=i$ is enclosed $\displaystyle \left.\frac{\cancel{(z+i)}\sqrt{z}}{\cancel{(z+i)}(z+i)}\right|_{z=i} = \frac{e^{i\pi/4}}{2i} =$

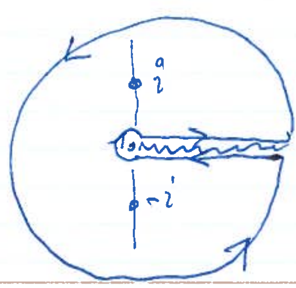
$$= \frac{\cos\pi/4 + i\sin\pi/4}{2i} = \frac{1+i}{2\sqrt{2}\,i}$$

$$\Rightarrow I = \frac{2\pi \cancel{i}}{\cancel{1+i}}\cdot\frac{\cancel{1+i}}{2\sqrt{2}\,\cancel{i}} = \frac{\pi}{\sqrt{2}}$$

<u>Solution 2:</u> $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = \operatorname{Re}\int_{-\infty}^{\infty}\frac{\sqrt{x}\,dx}{1+x^2} =$

(same contour as above) $\displaystyle = \operatorname{Re}\{2\pi i \sum \text{residues}\} = \operatorname{Re}\left\{\frac{2\pi \cancel{i}}{2\cancel{i}}\, e^{i\pi/4}\right\} = \pi\cos\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$

<u>Solution 3:</u> Introduce a branch cut $(0,\infty)$



The large circle gives zero contribution since

$$\left|\frac{z\sqrt{z}}{1+z^2}\right| \xrightarrow[R\to\infty]{} \frac{1}{\sqrt{|z|}} \to 0$$

The small circle gives zero contribution since

$$\left|\frac{z\sqrt{z}}{1+z^2}\right| \xrightarrow[z\to 0]{} 0 \text{ as } |z|\to 0$$

1, continued. We have above the branch cut $(0,\infty)$ $\dfrac{\sqrt{x}\,dx}{1+x^2}$

and below $(\infty,0)$ $\dfrac{\sqrt{xe^{2\pi i}}\; e^{2\pi i}\,dx}{1+(xe^{2\pi i})^2} = \dfrac{e^{i\pi}\sqrt{x}\,dx}{1+x^2} = -\dfrac{\sqrt{x}\,dx}{1+x^2}$

$$\Rightarrow \int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} + \int_\infty^0 \left(-\frac{\sqrt{x}\,dx}{1+x^2}\right) = 2\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = 2\pi i \sum\{\text{enclosed residues}\}$$

simple poles at $z=\pm i$ with residues

$$z=i: \quad \left.\frac{\cancel{(z-i)}\sqrt{z}}{\cancel{(z-i)}(z+i)}\right|_{z=i} = \frac{\sqrt{i}}{2i} = \frac{e^{i\pi/4}}{2i}$$

$$z=-i: \quad \left.\frac{\cancel{(z+i)}\sqrt{z}}{(z-i)\cancel{(z+i)}}\right|_{z=-i} = \frac{\sqrt{-i}}{-2i} = -\frac{e^{i3\pi/4}}{2i}$$

$$\therefore \int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = \pi\cdot i\left\{\frac{e^{i\pi/4}-e^{i3\pi/4}}{2i}\right\} = -\pi i e^{i\pi/2}\left\{\frac{e^{i\pi/4}-e^{-i\pi/4}}{2i}\right\} =$$

$$= \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

2. Use the Cauchy integral formula to construct a function $f(z)$ satisfying the properties:

(a) $f(z)$ is analytic except for a simple pole of residue R at $z=a$ and a branch cut $(0,\infty)$ at which the function has a discontinuity $f(x+i\varepsilon)-f(x-i\varepsilon)=2i\pi g(x)$, $x\geq 0$.

(b) $|f(z)|\to 0$ as $|z|\to\infty$, and $|zf(z)|\to 0$ as $|z|\to 0$.

Be careful with the specification of contours used and express the function in terms of R, a, and $g(x)$. (3p)

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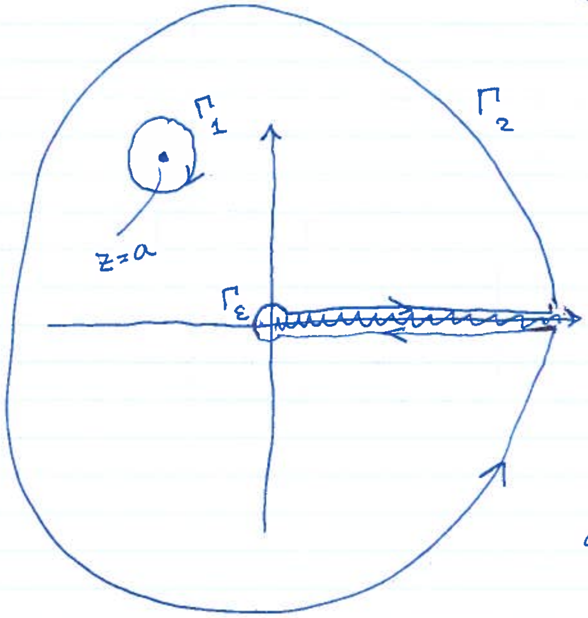
2. Use the Cauchy integral formula with contour given as



We have $f(z)=\dfrac{1}{2\pi i}\displaystyle\oint_{\Gamma_1}\frac{f(z')dz'}{z'-z}$ +

$$+\frac{1}{2\pi i}\int_0^\infty \frac{f(x'+i\varepsilon)-f(x'-i\varepsilon)}{x'-z}dx'$$

The integrals over the large circle Γ_2 and the small circle Γ_ε do not contribute due to properties (b).

The integral over Γ_1 (going clock wise)

$$\frac{1}{2\pi i}\oint_{\Gamma_1}\frac{f(z')}{z'-z}dz' = -\frac{R}{a-z} = \frac{R}{z-a}$$

From (a) we have $f(x'+i\varepsilon)-f(x'-i\varepsilon)=2\pi i g(x')$

Thus, $$f(z)=\frac{R}{z-a}+\int_0^\infty\frac{g(x')dx'}{x'-z}$$

3. Use the Frobenius method to find the two independent solutions to the equation (one sums to a simple function, the other may be given through the first four terms in the series):

$$3x\frac{d^2y}{dx^2}+(3x+1)\frac{dy}{dx}+y=0 \qquad \text{(4p)}$$

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3. $3xy'' + (3x+1)y' + y = 0$

Frobenius ansatz: $y = x^{\lambda}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+\lambda}$, $a_0 \neq 0$

$y' = \sum_{k=0}^{\infty} a_k (k+\lambda) x^{k+\lambda-1}$; $y'' = \sum_{k=0}^{\infty} a_k (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$

Equation

$$3\sum_{k=0}^{\infty} a_k (k+\lambda)(k+\lambda-1)x^{k+\lambda-1} + 3\sum_{k=0}^{\infty} a_k (k+\lambda)x^{k+\lambda} + \sum_{k=0}^{\infty} a_k (k+\lambda)x^{k+\lambda-1} +$$
$$+ \sum_{k=0}^{\infty} a_k x^{k+\lambda} = 0$$

Lowest order $x^{\lambda-1}$: $\{3a_0\lambda(\lambda-1) + a_0\lambda\}x^{\lambda-1} = 0$

$3\lambda^2 - 3\lambda + \lambda = 0 \Rightarrow 3\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - \frac{2}{3}) = 0$, $\lambda_i = \begin{cases} 0 \\ \frac{2}{3} \end{cases}$

Put together ($k \to k+1$ in the first and third sums)

$$\sum_{k=0}^{\infty}\left\{a_{k+1}\left[3(k+1+\lambda)(k+\lambda) + (k+1+\lambda)\right] + a_k\left[(k+\lambda)3 + 1\right]\right\}x^{k+\lambda} = 0$$

Each coefficient must be zero ;

$$a_{k+1} = -\frac{3(k+\lambda)+1}{(k+\lambda+1)(3(k+\lambda)+1)}a_k = -\frac{a_k}{k+\lambda+1}$$

$\lambda = 0$: $a_{k+1} = -\dfrac{a_k}{k+1}$, $a_1 = -a_0$, $a_2 = -\frac{1}{2}a_1 = \frac{1}{2\cdot 1}a_0$

$a_3 = -\frac{1}{3}a_2 = \frac{-1}{3\cdot 2\cdot 1}a_0$

$$a_n = \frac{(-1)^n}{n!}a_0$$

and $y(x) = a_0\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^k = a_0 e^{-x}$

Check: $y'(x) = -a_0 e^{-x} = -y$, $y''(x) = a_0 e^{-x} = y \Rightarrow 3xy - 3xy - y + y = 0$

ok

. 3 cont'd

$$\lambda = \frac{2}{3} \;: \quad a_{k+1} = -\frac{a_k}{k+\frac{5}{3}} \quad \text{or} \quad a_{k+1} = -\frac{3}{3k+5}\, a_k$$

$$a_1 = -\frac{3}{5} a_0 \,, \quad a_2 = -\frac{3}{8} a_1 = \frac{3^2}{8.5} a_0$$

$$a_3 = -\frac{3}{11} a_2 = -\frac{3^3}{11\cdot 8\cdot 5} a_0 \,, \quad a_4 = -\frac{3}{14} a_3 = \frac{3^4}{14\cdot 11\cdot 8\cdot 5} a_0$$

$$\therefore \quad y_2(x) = a_0 x^{2/3} \left\{ 1 - \frac{3}{5} x + \frac{3^2}{5\cdot 8} x^2 - \frac{3^3}{5\cdot 8\cdot 11} x^3 + \frac{3^4}{5\cdot 8\cdot 11\cdot 14} x^4 - \cdots \right\}$$

4. Find the Fourier transform of $f(x) = \begin{cases} 0, & x < 0 \\ e^{-ax}, & x > 0 \end{cases}$ where a is real and $a > 0$ (3p)

4. $$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\omega x} = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x(a-i\omega)}\, dx = \frac{1}{\sqrt{2\pi}} \left[- \frac{e^{-x(a-i\omega)}}{a-i\omega} \right]_0^{\infty} =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a-i\omega} = \frac{1}{\sqrt{2\pi}} \cdot \frac{a+i\omega}{a^2+\omega^2}$$

(Note: I had intended to replace this with a slightly more complicated transform, but forgot ...)

5. Use Laplace transforms to solve $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+13y=20e^{-t}$, $y(0)=1, y'(0)=3$. (4p)

5. Use Laplace transforms to solve

$$y''+4y'+13y = 20e^{-t} \quad y(0)=1,\ y'(0)=3$$

A&W 15.9 $\mathcal{L}\{F^{(2)}(t)\} = s^2\tilde{f} - sF(0) - F'(0)$

$\mathcal{L}\{F'(t)\} = s\tilde{f} - F(0)$

The transformed equation becomes $\qquad \mathcal{L}\{e^{kt}\} = \frac{1}{s-k}, s>k$ (A&W 15.103)

$$s^2\tilde{y} - s - 3 + 4s\tilde{y} - 4 + 13\tilde{y} = \frac{20}{s+1}$$

$$\tilde{y}(s^2+4s+13) = \frac{20}{s+1} + s+7$$

$$\tilde{y} = \frac{1}{s^2+4s+13}\left\{\frac{20}{s+1} + s+7\right\} = \frac{20+(s+7)(s+1)}{(s+1)(s^2+4s+13)} = \frac{s^2+8s+27}{(s+1)(s^2+4s+13)}$$

Partial fractions $\frac{s^2+8s+27}{(s+1)(s^2+4s+13)} = \frac{a}{s+1} + \frac{bs+c}{s^2+4s+13} = \cancel{as^2+4as+13a+bs^2+}$

$$= \frac{as^2+4as+13a+bs^2+bs+cs+c}{(s+1)(s^2+4s+13)}$$

s^2: $a+b=1 \rightarrow b = 1-a$

s: $4a+b+c=8 \rightarrow 4a+1-a+c=8 \rightarrow 3a+c=7$

s^0: $13a+c=27 \rightarrow c = 27-13a$

$3a+27-13a=7$

$-10a = -20$

$\therefore a=2,\ b=-1,\ c=1$

$$\tilde{y} = \frac{2}{s+1} + \frac{1-s}{(s+2)^2+3^2} = \frac{2}{s+1} + \frac{3}{(s+2)^2+3^2} - \frac{s+2}{(s+2)^2+(3)^2}$$

From tables and shift by substitution

$$y(t) = 2e^{-t} + e^{-2t}\sin 3t - e^{-2t}\cos 3t$$

Bromwich Integrals on next page

Using Bromwich integrals:

$$\mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} = \frac{1}{\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} \frac{e^{st}}{s+1}\,ds = \frac{1}{\pi i}\oint \frac{e^{zt}\,dz}{z+1} = 2\left\{\text{res } z=-1\right\} = 2e^{-t}$$

closing in left half-plane (LHP)

$$\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\} = \frac{3}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} \frac{e^{st}}{(s+2)^2+3^2}\,ds = \frac{3}{2\pi i}\oint_{LHP} \frac{e^{zt}}{(z+2)^2+3^2}\,dz = 3\sum\left\{\text{residues}\right\}$$

$$(z+2)^2+3^2=0 \quad \Rightarrow \quad z=-2\pm 3i$$

$$\text{Res } z=-2+3i: \quad \left.\frac{\cancel{(z+2-3i)}\,e^{zt}}{\cancel{(z+2-3i)}(z+2+3i)}\right|_{z=-2+3i} = \frac{e^{-2t}\cdot e^{i3t}}{6i}$$

$$\text{Res } z=-2-3i: \quad \left.\frac{\cancel{(z+2+3i)}\,e^{zt}}{(z+2-3i)\cancel{(z+2+3i)}}\right|_{z=-2-3i} = \frac{e^{-2t}\cdot e^{-i3t}}{-6i}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\} = 3e^{-2t}\left\{\frac{e^{i3t}-e^{-i3t}}{6i}\right\} = e^{-2t}\sin 3t$$

$$\mathcal{L}^{-1}\left\{-\frac{s+2}{(s+2)^2+3^2}\right\} = -\frac{1}{2\pi i}\oint_{LHP} \frac{(z+2)e^{zt}}{(z+2)^2+3^2}\,dz = -\sum \text{residues}$$

$$\text{Res } z=-2+3i: \quad \left.\frac{\cancel{(z+2-3i)}(z+2)e^{zt}}{\cancel{(z+2-3i)}(z+2+3i)}\right|_{z=-2+3i} = \frac{3i\,e^{-2t}\cdot e^{i3t}}{6i}$$

$$\text{Res } z=-2-3i: \quad \left.\frac{\cancel{(z+2+3i)}(z+2)e^{zt}}{(z+2-3i)\cancel{(z+2+3i)}}\right|_{z=-2-3i} = \frac{-3i\,e^{-2t}\,e^{-i3t}}{-6i}$$

$$\therefore \mathcal{L}^{-1}\left\{-\frac{s+2}{(s+2)^2+3^2}\right\} = -e^{-2t}\left\{\frac{e^{i3t}+e^{-i3t}}{2}\right\} = -e^{-2t}\cos 3t$$

$$\therefore y(t) = 2e^{-t} + e^{-2t}\sin 3t - e^{-2t}\cos 3t$$

6. Starting from the recursion relation $(2l+1)xP_l = lP_{l-1} + (l+1)P_{l+1}$ for the Legendre polynomials derive the recursion relation for the associated Legendre polynomials:
$(l+1-m)P_{l+1}^m = (2l+1)xP_l^m - (l+m)P_{l-1}^m$ (2p)

6. Use $(2\ell+1)xP_\ell = \ell P_{\ell-1} + (\ell+1)P_{\ell+1}$ (1)

$$(2\ell+1)P_\ell = P'_{\ell+1} - P'_{\ell-1} \quad (2)$$

and $P_\ell^m = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell$

Take m:th derivative of (1) where $\frac{d^m}{dx^m}(xP_\ell) = x\frac{d^m}{dx^m}P_\ell + m\frac{d^{m-1}}{dx^{m-1}}P_\ell$

$$\Rightarrow (2\ell+1)x\frac{d^m}{dx^m}P_\ell + (2\ell+1)m\frac{d^{m-1}}{dx^{m-1}}P_\ell = \ell\frac{d^m}{dx^m}P_{\ell-1} + (\ell+1)\frac{d^m}{dx^m}P_{\ell+1}$$

$$(2\ell+1)\frac{d^{m-1}}{dx^{m-1}}P_\ell = \frac{d^{m-1}}{d^{m-1}}\left\{P'_{\ell+1} - P'_{\ell-1}\right\} = \frac{d^m}{dx^m}P_{\ell+1} - \frac{d^m}{dx^m}P_{\ell-1}$$

$$\therefore \quad (2\ell+1)x\frac{d^m}{dx^m}P_\ell + m\frac{d^m}{dx^m}P_{\ell+1} - m\frac{d^m}{dx^m}P_{\ell-1} = \ell\frac{d^m}{dx^m}P_{\ell-1} + (\ell+1)\frac{d^m}{dx^m}P_{\ell+1}$$

scale by $(1-x^2)^{m/2}$

$$(\ell+1-m)P_{\ell+1}^m = (2\ell+1)xP_\ell^m - (\ell+m)P_{\ell-1}^m$$

7. Solve $\left(1+x^{2}\right)\dfrac{dy}{dx}+6xy=2x$, $x\in R$ (3p)

7. Use variation of the constant to find the solution

Put equation on regular form

$$y' + \frac{6x}{1+x^2}\,y = \frac{2x}{1+x^2}$$

First solve the homogeneous equation

$$\frac{y_0'}{y_0} = -\frac{6x}{1+x^2} \Rightarrow \ln y_0 = -3\int^{x} \frac{2x'dx'}{1+x'^2} + \ln C$$

$$\ln y_0(x) = -3\ln(1+x^2) + \ln C$$

$$\Rightarrow y_0(x) = \frac{C}{(1+x^2)^3}$$

Particular Solution: $C \to C(x)$

$$y' = -\frac{6x\,C}{(1+x^2)^4} + \frac{C'(x)}{(1+x^2)^3} = -\frac{6x}{1+x^2}\,y + \frac{C'(x)}{(1+x^2)^3}$$

$$\Rightarrow \frac{C'(x)}{(1+x^2)^3} = \frac{2x}{1+x^2}$$

$$C'(x) = 2x(1+x^2)^2$$

$$\Rightarrow C(x) = \frac{1}{3}(1+x^2)^3$$

$$y_p(x) = \frac{1}{3}\,\frac{(1+x^2)^3}{(1+x^2)^3} = \frac{1}{3}$$

$$y(x) = \frac{C}{(1+x^2)^3} + \frac{1}{3}$$

Test solution: $y'(x) = -\dfrac{6x\,C}{(1+x^2)^4}$

$$(1+x^2)y' + 6xy = -\frac{6x\,C}{(1+x^2)^3} + \frac{6x\,C}{(1+x^2)^3} + \frac{1}{3}\cdot 6x = 2x \quad \text{OK}$$

8. Use the Laplace transform technique and the defining equation of the Green's function to find the Green's function for the equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = f(t)$, where $y(0) = y'(0) = 0$ and $y(t) = 0$ for $t < 0$. (4p)

8. $y'' + 2y' + y = f(t) \qquad y(0) = y'(0) = 0 \qquad y(t) = 0$ for $t < 0$

Green's function : $G'' + 2G' + G = \delta(t - t')$ satisfies the same boundary conditions

$\mathcal{L}\{G'\} = s\tilde{g} - G(0) = s\tilde{g}$

$\mathcal{L}\{G''\} = s^2\tilde{g} - sG(0) - G'(0) = s^2\tilde{g}$

$$\mathcal{L}\{\delta(t-t')\} = \int_0^\infty e^{-st}\delta(t-t')\,dt = e^{-st'}$$

The equation becomes $(s^2 + 2s + 1)\tilde{g} = e^{-st'}$

$$\tilde{g}(s) = \frac{e^{-st'}}{s^2+2s+1} = \frac{e^{-st'}}{(s+1)^2}$$

Bromwich integral $G(t,t') = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} \frac{e^{-st'} \cdot e^{st}}{(s+1)^2}\,ds =$

$$= \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} \frac{e^{s(t-t')}}{(s+1)^2}\,ds$$

For $t > t'$ we can close in the left half-plane and apply Jordan's lemma

$$\frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} \frac{e^{s(t-t')}}{(s+1)^2}\,ds = \frac{1}{2\pi i}\oint \frac{e^{z(t-t')}}{(z+1)^2}\,dz = \sum \text{residues}$$

The double pole at $z = -1$ is included

$$\text{Res}\{z=-1\} = \frac{d}{dz}\left.\frac{e^{z(t-t')}\cancel{(z+1)^2}}{\cancel{(z+1)^2}}\right|_{z=-1} = (t-t')e^{z(t-t')}\Big|_{z=-1} =$$

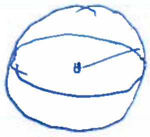
$$= (t-t')e^{-(t-t')}$$

For $t < t'$ no poles are included $\rightarrow G(t,t') = 0$

$$\therefore \quad G(t,t') = \begin{cases} 0, & t < t' \\ (t-t')e^{-(t-t')}, & t > t' \end{cases}$$

9. A sphere initially at 0° has its surface kept at 100° from $t=0$ on (for example a frozen pea in boiling water). Find the time-dependent temperature distribution. (Hint: Subtract 100° from all temperatures and solve the problem; then add the 100° to the answer). (5p)

9.



Assume radius r_0 ; initial temperature $T_0 = 0°$

From time $t=0$ the surface is kept at $100°$

To have homogeneous boundary conditions we take

$u(r,\vartheta,\varphi,t) = 100 + y(r,\vartheta,\varphi,t)$ such that

$y(r_0,\vartheta,\varphi,t) = 0°$ as boundary cond. which makes

$u(r,\vartheta,\varphi,t)$ satisfy the real boundary condition.

Spherical polar coordinates. The heat equation separates into spatial and temporal parts through $y(r,\vartheta,\varphi,t) = R(r)\Theta(\vartheta)\Phi(\varphi)T(t)$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{R\Theta\Phi}\nabla^2[R(r)\Theta(\vartheta)\Phi(\varphi)] = -k^2$$

Time-dependence: $\frac{dT}{dt} + k^2 T = 0 \rightarrow T(t) = e^{-k^2 t}$

The spatial dependence separates into (A&W section 9.3)

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0$$ No φ-dependence $\rightarrow m = 0$

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{d\Theta}{d\vartheta}\right) - \frac{m^2}{\sin^2\vartheta}\Theta + Q\Theta = 0$$ No ϑ-dependence $\rightarrow Q = 0$ (i.e. $Y_0^0(\vartheta,\varphi)$)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2 R - \frac{QR}{r^2} = 0$$

Radial equation $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2 r^2 R = 0$ spherical Bessel eq. for $j_0(kr)$ (*)

(*) spherical Neumann functions excluded since origin is included

The boundary condition at r_0 implies $kr_0 = \alpha_{0\nu}$ a zero of j_0

i.e. $k = \frac{\alpha_{0\nu}}{r_0}$ and $y_\nu(r,\vartheta,\varphi,t) = y_\nu(r,t) = j_0\left(\alpha_{0\nu}\frac{r}{r_0}\right)e^{-\left(\frac{\alpha_{0\nu}}{r_0}\right)^2 t}$

$$j_0(x) = \frac{\sin x}{x} \Rightarrow \alpha_{0\nu} = n\pi$$

• 9, cont'd and $y_n(r,t) = \frac{1}{r} \sin\left(n\pi \frac{r}{r_0}\right) e^{-\gamma\left(\frac{n\pi}{r_0}\right)^2 t}$

Initial steady state $y(r, t\leq 0) = -100$

$$-100 = \sum a_n j_0\left(n\pi \frac{r}{r_0}\right)$$

Project

$$-100 \int_0^{r_0} j_0\left(m\pi \frac{r}{r_0}\right) r^2 dr = \sum_n a_n \underbrace{\int_0^{r_0} j_0\left(m\pi\frac{r}{r_0}\right) j_0\left(n\pi \frac{r}{r_0}\right) r^2 dr}_{\frac{r_0^3}{2}\left[j_1(n\pi)\right]^2 \delta_{nm}}$$

$$\int_0^{r_0} j_0\left(m\pi \frac{r}{r_0}\right) r^2 dr = \begin{bmatrix} x = m\pi \frac{r}{r_0} \\ r = \frac{x r_0}{m\pi} \\ dr = \frac{r_0}{m\pi} dx \\ r = r_0 \rightarrow x = m\pi \end{bmatrix} = \left(\frac{r_0}{m\pi}\right)^3 \int_0^{m\pi} j_0(x) x^2 dx =$$

$$= \left(\frac{r_0}{m\pi}\right)^3 \int_0^{m\pi} \frac{d}{dx}\left[x^2 j_1(x)\right] dx.$$

$$= \left(\frac{r_0}{m\pi}\right)^3 \left[x^2 J_1(x)\right]_0^{m\pi} = \frac{r_0^3}{m\pi} j_1(m\pi) = -\frac{r_0^3}{m\pi}\frac{\cos(m\pi)}{m\pi} =$$

$$= \frac{r_0^3}{(m\pi)^2}(-1)^{m+1} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$\therefore \quad -100 \cdot \frac{r_0^3}{(m\pi)^2}(-1)^{m+1} = \frac{a_m r_0^3}{2}\left[j_1(m\pi)\right]^2 =$$

$$= \frac{a_m r_0^3}{2}\left[-\frac{\cos m\pi}{m\pi}\right]^2 = \frac{a_m r_0^3}{2} \cdot \frac{1}{(m\pi)^2}$$

$$\therefore \quad a_m = 200(-1)^m$$

Thus $y(r,t) = 100 + 200 \sum_{n=0}^{\infty} (-1)^n j_0\left(n\pi \frac{r}{r_0}\right) e^{-\gamma\left(\frac{n\pi}{r_0}\right)^2 t}$