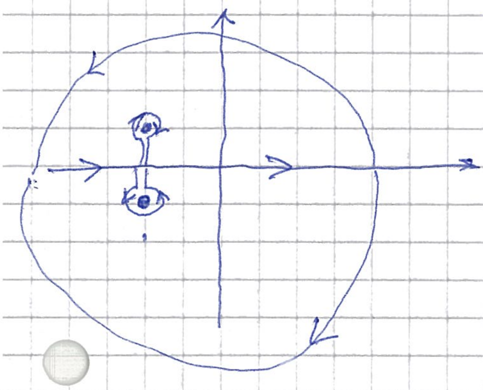
1. Use calculus of residues to evaluate the integral $\int_{-\infty}^{\infty} \frac{x\sin(\pi x)}{x^2+4x+5}\,dx$. (3p)

Solution: Rewrite $\sin \pi x = \frac{1}{2i}\left(e^{i\pi x} - e^{-i\pi x}\right)$ and apply Jordan's lemma to suitably chosen contours.

$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+4x+5}\,dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2+4x+5}\,dx - \frac{1}{2i}\int_{-\infty}^{\infty} \frac{x e^{-i\pi x}}{x^2+4x+5}\,dx = I_1 - I_2$$

We have poles at $x^2+4x+5=0 \Rightarrow x=-2\pm i$

Extend to the complex plane. Since in I_1 the exponential is positive and the integrand goes to zero as $\frac{1}{z}$ we can apply Jordan's lemma in the upper half-plane without the added half-circle contributing and containing $z=-2+i$ as pole



$$I_1 = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{z e^{i\pi z}\,dz}{z^2+4z+5} = \frac{2\pi i}{2i}\,\mathrm{Res}\{z=-2+i\} = \pi\,\mathrm{Res}\{z=-2+i\}$$

$$\mathrm{Res}\{z=-2+i\} = \left.\frac{\cancel{(z+2-i)}\, z e^{i\pi z}}{\cancel{(z+2-i)}(z+2+i)}\right|_{z=z_+} = \frac{(-2+i)e^{i\pi(-2+i)}}{2i}$$

$$= \frac{(i-2)}{2i}e^{-2\pi i}\cdot e^{-\pi} = \frac{(i-2)}{2i}e^{-\pi}$$

$$\therefore I_1 = \frac{(i-2)\pi e^{-\pi}}{2i}$$

For I_2 we must close in the lower half-plane and the resulting contour goes clockwise so that

$$I_2 = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{z e^{i\pi z}\,dz}{z^2+4z+5} = -\frac{2\pi i}{2i}\,\mathrm{Res}\{z=-2-i\} = -\pi\,\mathrm{Res}\{z=-2-i\}$$

$$\mathrm{Res}\{z=-2-i\} = \frac{(-2-i)e^{-i\pi(-2-i)}}{-2i} = \frac{i+2}{2i}e^{-\pi}$$

Finally,
$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+4x+5}\,dx = I_1 - I_2 = \pi e^{-\pi}\left\{\frac{i-2}{2i} + \frac{i+2}{2i}\right\} = \pi e^{-\pi}$$

Equivalent $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+4x+5}\,dx = \mathrm{Im}\left\{\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2+4x+5}\,dx\right\}$

2. Express the polynomial $P(x) = 7x^4 - 3x + 1$ as a linear combination of Legendre polynomials.

Since the highest power is x^4 we need not go higher than P_4. Let's list the first five polynomials:

$$P_0 = 1 \qquad P_3 = \frac{1}{2}(5x^3 - 3x)$$

$$P_1 = x \qquad P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_2 = \frac{1}{2}(3x^2 - 1)$$

From P_4 we get $35x^4 = 8P_4 + 30x^2 - 3 \Rightarrow 7x^4 = \frac{8}{5}P_4 + 6x^2 - \frac{3}{5}$

so that $P(x) = \frac{8}{5}P_4 + 6x^2 - \frac{3}{5} - 3x + 1 = \frac{8}{5}P_4 + 6x^2 - 3x + \frac{2}{5}$

Next highest power is x^2 which corresponds to $P_2 = \frac{1}{2}(3x^2 - 1)$ which gives $x^2 = \frac{2}{3}P_2 + \frac{1}{3}$

$$\Rightarrow P(x) = \frac{8}{5}P_4 + 6\left(\frac{2}{3}P_2 + \frac{1}{3}\right) - 3x + \frac{2}{5} = \frac{8}{5}P_4 + 4P_2 + 2 - 3x + \frac{2}{5} =$$

$$= \frac{8}{5}P_4 + 4P_2 - 3P_1 + \frac{12}{5}P_0$$

3. Use the Frobenius method to find the two independent solutions to the equation (Note: the solutions sum to simple closed forms): $x^2y'' + 4xy' + (x^2+2)y = 0$ (4p)

3. Use Frobenius to find the two solutions to

$$x^2y'' + 4xy' + (x^2+2)y = 0$$

Ansatz: $y = x^k \sum_{p=0}^{\infty} a_p x^p \quad , a_0 \neq 0$

$$y' = \sum_{p=0}^{\infty} a_p (k+p) x^{p+k-1}$$

$$y'' = \sum_{p=0}^{\infty} a_p (k+p)(k+p-1) x^{k+p-2}$$

The equation: $\sum_{p=0}^{\infty} a_p (k+p)(k+p-1) x^{k+p} + 4 \sum_{p=0}^{\infty} a_p (k+p) x^{k+p} +$

$$+ 2 \sum_{p=0}^{\infty} a_p x^{k+p} + \sum_{p=0}^{\infty} a_p x^{k+p+2} = 0$$

Lowest power: $x^k \Rightarrow a_0 \{ k(k-1) + 4k + 2 \} = 0$

$\Rightarrow k^2 + 3k + 2 = 0 \qquad k_1 = -2,$
$k_2 = -1$

Take $k=-1$: For the next power (x^{k+1} ~~p=1~~) we have

$$a_1 \{ (k+1)k + 4(k+1) + 2 \} = 0 \Rightarrow 2a_1 = 0$$

Recursion relation: $a_p + a_{p+2} \{ (k+p+2)(k+p+1) + 4(k+p+2) + 2 \} = 0$

shift $p \to p-2$

$$a_{p-2} + a_p \{ (k+p)(k+p-1) + 4(k+p) + 2 \} = 0$$

$k=-1$ $\qquad a_p = -\frac{a_{p-2}}{(p-1)(p-2) + 4(p-1) + 2} = -\frac{a_{p-2}}{p(p+1)}$

3 continued

Only even terms

$$a_2 = -\frac{a_0}{3!}, \quad a_4 = \frac{a_0}{5!}, \quad a_6 = -\frac{a_0}{7!}$$

$$y(x) = \frac{a_0}{x} \sum_{p=0}^{\infty} (-1)^p \frac{1}{(2p+1)!} x^{2p} = \frac{a_0}{x^2} \sum_{p=0}^{\infty} (-1)^p \frac{1}{(2p+1)!} x^{2p+1} = a_0 \frac{\sin x}{x^2}$$

Take $k=-2$; Take $a_1 = 0$ as before. Since the equation allows odd and even solutions.

We get $a_{p-2} + a_p \{(p-2)(p-3) + 4(p-2) + 2\} = 0$

and $a_p = -\frac{a_{p-2}}{p(p-1)}$

$$a_2 = -\frac{a_0}{2!}, \quad a_4 = \frac{a_0}{4!}, \quad a_6 = -\frac{a_0}{6!}$$

$$y(x) = \frac{a_0}{x^2} \sum_{p=0}^{\infty} (-1)^p \frac{1}{(2p)!} x^{2p} = a_0 \frac{\cos x}{x^2}$$

4. Find the inverse Laplace transform of $F(s)=\dfrac{5}{(s+2)(s^2+1)}$ by solving the Bromwich integral. (3p)

The inverse $\mathcal{L}^{-1}\{F(s)\} = \dfrac{5}{2\pi i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{(s+2)(s^2+1)}\,ds = 5\sum$ enclosed residues

We have poles at $s=-2, i, -i$

$$\text{Res}\{s=-2\} = \left.\frac{\cancel{(s+2)}e^{st}}{\cancel{(s+2)}(s^2+1)}\right|_{s=-2} = \frac{e^{-2t}}{5}$$

$$\text{Res}\{s=i\} = \left.\frac{\cancel{(s-i)}e^{st}}{(s+2)\cancel{(s-i)}(s+i)}\right|_{s=i} = \frac{e^{it}}{2i(2+i)}$$

$$\text{Res}\{s=-i\} = \left.\frac{\cancel{(s+i)}e^{st}}{(s+2)(s-i)\cancel{(s+i)}}\right|_{s=-i} = \frac{e^{-it}}{(-2i)(2-i)}$$

$$\frac{1}{2i}\left\{\frac{e^{it}}{2+i} - \frac{e^{-it}}{2-i}\right\} = \frac{1}{5}\cdot\frac{1}{2i}\left\{(2-i)e^{it} - (2+i)e^{-it}\right\} =$$

$$= \frac{1}{5}\left[2\cdot\frac{1}{2i}\left(e^{it}-e^{-it}\right) - \frac{i}{2i}\left(e^{it}+e^{-it}\right)\right] =$$

$$= \frac{1}{5}\left(2\sin t - \cos t\right)$$

$$\therefore\ \mathcal{L}^{-1}\{F(s)\} = e^{-2t} + 2\sin t - \cos t$$

5. An elastic string is fixed between the two points x=0 and x=L. The string is deformed according to $u(x,0)=x(x-L)$ and then released at time t=0. This determines the necessary initial conditions for a determination of the subsequent motion. Find $u(x,t)$ describing the amplitude of oscillations in this motion! (4p)

5. The initial conditions are

$$\begin{cases} u(x,0) = x(x-L) \\ u_t'(x,0) = 0 \end{cases} \quad \text{and } u(0,t) = u(L,t) = 0$$

(the string is held at rest and then released)

The wave equation $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ separates with the ansatz

$u(x,t) = X(x)\,T(t)$ to give

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = \frac{1}{c^2 T(t)}\frac{d^2T}{dt^2} = -\lambda^2$$

$$\begin{cases} \frac{d^2X}{dx^2} = -\lambda^2 X(x) & \Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x \\ \frac{d^2T}{dt^2} = -c^2\lambda^2 T(t) & \Rightarrow T(t) = C\cos(\lambda ct) + D\sin(\lambda ct) \end{cases}$$

Boundary conditions $u(0,t) = u(L,t) = 0$ gives

$X(0) = 0 \Rightarrow A = 0$

$X(L) = 0 \Rightarrow B\sin\lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L}$ n integer

$\therefore X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right)$

$T_n(t) = C_n\cos\omega_n t + D_n \sin\omega_n t$ where $\omega_n = \frac{n\pi c}{L}$

General solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left\{C_n\cos\omega_n t + D_n\sin\omega_n t\right\}$$

Introducing the initial value for the velocity we have

$$\left\{-\omega_n C_n \sin\omega_n t + \omega_n D_n\cos\omega_n t\right\}_{t=0} = D_n\omega_n = 0$$

So that $u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t$

5, continued

At $t=0$ we have $u(x,0) = x(x-L) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$

The coefficient c_m is found by projection where $\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}\delta_{nm}$

Thus, $$c_m = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi}{L}x\right)x(x-L)\,dx = \frac{2}{L}\int_0^L \left\{x^2 \sin\left(\frac{m\pi}{L}x\right) - xL\sin\left(\frac{m\pi}{L}x\right)\right\}dx =$$

$$= \frac{2}{L}\int_0^L \left\{-\left(\frac{L}{\pi}\right)^2 \frac{d^2}{dm^2}\sin\left(\frac{m\pi}{L}x\right) + \frac{L^2}{\pi}\frac{d}{dm}\cos\left(\frac{m\pi}{L}x\right)\right\}dx =$$

$$= \frac{2}{L}\left(\frac{L}{\pi}\right)^2\left\{-\frac{d^2}{dm^2}\int_0^L \sin\frac{m\pi}{L}x\,dx\right\} + 2\frac{L}{\pi}\frac{d}{dm}\underbrace{\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx}_{=0} =$$

$$= -\frac{2L}{\pi^2}\frac{d^2}{dm^2}\left[-\frac{L}{m\pi}\cos\left(\frac{m\pi}{L}x\right)\right]_0^L = \frac{2L^2}{\pi^3}\frac{d^2}{dm^2}\left(\frac{1}{m}\right)\left(\cos(m\pi) - 1\right) =$$

$$= \frac{4L^2}{\pi^3 m^3}\left(\cos(m\pi) - 1\right) = \begin{cases} 0, & m \text{ even} \\ -\frac{8L^2}{\pi^3 m^3}, & m \text{ odd} \end{cases}$$

$$\Rightarrow \quad u(x,t) = \frac{4L^2}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t}{n^3}\left(\cos(n\pi) - 1\right)$$

6. The current, $I(t)$, through an RLC circuit (resistance R, inductance L and capacitance C) is given by the equation $L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=f(t)$.

Use the Fourier transform $\tilde{g}(\omega,t')=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G(t,t')\exp(i\omega t)dt$ to show that the retarded Green function $G_r(t,t')$ is given by $G_r(t,t')=\begin{cases}0 & t<t' \\ \omega_1^{-1}\exp(-\beta(t-t'))\sin(\omega_1(t-t')) & t'<t\end{cases}$

where $\omega_1=\sqrt{\omega_0^2-\beta^2}$, $\beta=\frac{R}{2L}$ and $\omega_0=\frac{1}{\sqrt{LC}}$. Retarded Green function means that the effect at time t comes after the cause at time t', *i.e.* $t'<t$. (4p)

6. The equation is $L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=f(t)$ with $f(t)=\begin{cases}0, & t<0 \\ \gamma\exp(-\lambda t), & t>0\end{cases}$

The Green function $L\ddot{G}+R\dot{G}+\frac{1}{C}G=\delta(t-t')$

Fourier transform $\left(-\omega^2L-i\omega R+\frac{1}{C}\right)\tilde{g}(\omega,t')=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t-t')e^{i\omega t}dt=\frac{e^{i\omega t'}}{\sqrt{2\pi}}$

solve for $\tilde{g}(\omega,t')$: $\tilde{g}(\omega)=\frac{1}{\sqrt{2\pi}}\frac{e^{i\omega t'}}{-\omega^2L-i\omega R+\frac{1}{C}}$

invert to find $G(t,t')$: $G(t,t')=-\frac{1}{2\pi L}\int_{-\infty}^{\infty}\frac{e^{i\omega(t'-t)}}{\omega^2+i\omega\frac{R}{L}-\frac{1}{CL}}d\omega$

We have poles when $\omega^2+i\omega\frac{R}{L}-\frac{1}{CL}=0\Rightarrow\left(\omega+\frac{iR}{2L}\right)^2=\frac{1}{CL}-\left(\frac{R}{2L}\right)^2$

$\therefore\ \omega=-i\beta\pm\sqrt{\omega_0^2-\beta^2}=-i\beta\pm\omega_1$

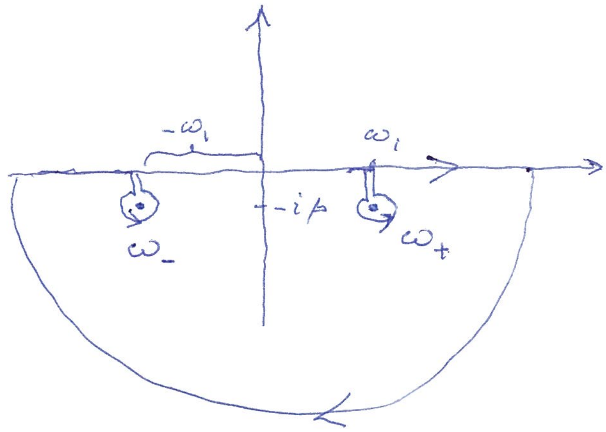
with $\beta=\frac{R}{2L}$, $\omega_0=\frac{1}{\sqrt{LC}}$

The integrand decays as $\frac{1}{\omega^2}$ so for $t<t'$ we can apply Jordan's lemma and close in the upper half plane. Both poles are in the lower half plane so

$G(t,t')=0$ for $t<t'$

For $t>t'$ we evaluate the residues at $\omega=-i\beta\pm\omega_1$

6. continued



$$\text{Res.}\{z=\omega_-\} =$$

$$= \frac{\cancel{(z-\omega_-)}\, e^{iz(t'-t)}}{\cancel{(z-\omega_-)}(z-\omega_+)}\Bigg|_{z=\omega_-}$$

$$= \frac{e^{i(-i\beta-\omega_1)(t'-t)}}{-i\beta-\omega_1+i\beta-\omega_1} = -\frac{1}{2\omega_1} e^{-\beta(t-t')} \cdot e^{+i\omega_1(t-t')}$$

$$\text{Res.}\{z=\omega_+\} = \frac{\cancel{(z-\omega_+)}\, e^{iz(t'-t)}}{(z-\omega_-)\cancel{(z-\omega_+)}}\Bigg|_{z=\omega_+} = \frac{e^{i(-i\beta+\omega_1)(t'-t)}}{-i\beta+\omega_1+i\beta+\omega_1} =$$

$$= \frac{1}{2\omega_1} e^{-\beta(t-t')} \cdot e^{-i\omega_1(t-t')}$$

By Cauchy's integral theorem we have $0 = -\int_{-\infty}^{\infty} + \int_{\omega_-} + \int_{\omega_+}$

which gives $G(t,t') = +2\pi i\,\{\text{Res}\{\omega_-\} + \text{Res}\{\omega_+\}\} =$

$$= +\frac{2\pi i}{2\pi L}\left\{ e^{-i\omega_1(t-t')} - e^{i\omega_1(t-t')} \right\} \frac{1}{2\omega_1} e^{-\beta(t-t')} =$$

$$= +\frac{1}{\omega_1 L} e^{-\beta(t-t')} \sin \omega_1(t-t')$$

7. A vibrating circular drum membrane is fixed at the boundary $r = r_0$, i.e. $u(r_0,\varphi,t)=0$. At time $t=0$ the deformation of the membrane is described by the function $u(r,\varphi,0)=f(r,\varphi)$ and the velocity $u_t'(r,\varphi,0)=g(r,\varphi)$. At all times the deformation of the membrane is finite. Determine the motion of the membrane by solving the associated wave equation $\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$. The normalization of the required orthogonal functions should be considered but need not be evaluated. (4p)

We have the initial and boundary conditions

$$u(r_0,\varphi,t)=0$$
$$|u(r,\varphi,t)|<\infty$$
$$u(r,\varphi,0)=f(r,\varphi)$$
$$u_t'(r,\varphi,0)=g(r,\varphi)$$

In addition the function has to be periodic in the angle φ

i.e. $u(r,\varphi+2\pi,t)=u(r,\varphi,t)$

The equation can be separated through the ansatz

$u(r,\varphi,t) = \psi(r,\varphi)\,T(t)$ giving

$$\frac{1}{\psi}\nabla^2\psi = \frac{1}{c^2}\frac{T''}{T} = -\varkappa^2$$

We have
$$\begin{cases} \nabla^2\psi + \varkappa^2\psi = 0 \\ T'' + c^2\varkappa^2 T = 0 \end{cases}$$

Use polar coordinates: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$

Separate r and φ through $\psi(r,\varphi)=R(r)\Phi(\varphi)$

$$\Phi\frac{d^2R}{dr^2} + \Phi\frac{1}{r}\frac{dR}{dr} + R\frac{1}{r^2}\frac{d^2\Phi}{d\varphi^2} + \varkappa^2 R\Phi = 0$$

$$\frac{1}{R}\left[r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \varkappa^2 r^2 R\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = \alpha^2$$

$$\Rightarrow \begin{cases} r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\varkappa^2 r^2 - \alpha^2)R = 0 \\ \frac{d^2\Phi}{d\varphi^2} + \alpha^2\Phi = 0 \quad \Rightarrow \Phi(\varphi) = Ae^{i\alpha\varphi} + Be^{-i\alpha\varphi} \end{cases}$$

$\Phi(\varphi+2\pi) = \Phi(\varphi) \Rightarrow \alpha = n$, integer

~~Take $\Phi(\varphi) = A_n\cos(n\varphi) + B_n\sin(n\varphi)$~~

$$\Rightarrow \Phi_n(\varphi) = A_n e^{in\varphi} + B_n e^{-in\varphi} = A_n\cos(n\varphi) + B_n\sin(n\varphi)$$

7, continued

The radial equation: $r^2R'' + rR' + (\varkappa^2 r^2 - n^2)R = 0$

can be converted into the Bessel equation through the scaling

$$x = \varkappa r, \quad \frac{d}{dr} = \frac{d}{dx}\frac{dx}{dr} = \varkappa\frac{d}{dx}, \quad \frac{d^2}{dr^2} = \varkappa^2\frac{d^2}{dx^2}$$

$$\Rightarrow \frac{x^2}{\varkappa^2}\varkappa^2\frac{d^2R}{dx^2} + \frac{x}{\varkappa}\varkappa\frac{dR}{dx} + (x^2 - n^2)R = 0$$

$$\Rightarrow x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0$$

Solutions $R_n(x) = c_n J_n(x) + d_n N_n(x)$

$|u(r,\varphi,t)| < \infty \Rightarrow d_n = 0$

$\therefore R_n(x) = J_n(x)$ or $R_n(r) = J_n(\varkappa r)$

The boundary condition $u(r_0,\varphi,t) = 0$ gives $J_n(\varkappa r_0) = 0$

i.e. $\varkappa_n r_0 = a_{ns}$ where a_{ns} is the s'th root of J_n.

$\therefore \varkappa_{ns} = \frac{a_{ns}}{r_0} \Rightarrow R_{ns}(r) = J_n\left(\frac{a_{ns}}{r_0}r\right)$

The equation for the time $T'' + c^2\varkappa^2 T = 0$

now gives $T(t) = c_{ns}\cos(\omega_{ns}t) + d_{ns}\sin(\omega_{ns}t)$, $\omega_{ns} = \frac{c\,a_{ns}}{r_0}$

We can now put together the full solution as

$$u(r,\varphi,t) = \sum_{n=-\infty}^{\infty}\sum_{s=1}^{\infty} J_n\left(a_{ns}\frac{r}{r_0}\right)\left\{A_n\cos(n\varphi) + B_n\sin(n\varphi)\right\}\left\{c_{ns}\cos(\omega_{ns}t) + d_{ns}\sin(\omega_{ns}t)\right\}$$

At $t=0$

$$u(r,\varphi,0) = f(r,\varphi) = \sum_{n=-\infty}^{\infty}\sum_{s=1}^{\infty} c_{ns}J_n\left(a_{ns}\frac{r}{r_0}\right)\left\{A_n\cos(n\varphi) + B_n(\sin(n\varphi)\right\}$$

Use the Bessel and trigonometric function orthogonality over the respective intervals:

$$c_{ms}\cdot\begin{Bmatrix}A_m\\B_m\end{Bmatrix}\int_0^{r_0}\left[J_m\left(a_{mq}\frac{r}{r_0}\right)\right]^2 r\,dr\cdot\int_0^{2\pi}\begin{Bmatrix}\cos^2(m\varphi)\\\sin^2(m\varphi)\end{Bmatrix}d\varphi =$$

$$= \int_0^{r_0}\int_0^{2\pi} J_m\left(a_{mq}\frac{r}{r_0}\right)\begin{Bmatrix}\cos(m\varphi)\\\sin(m\varphi)\end{Bmatrix} f(r,\varphi)\,r\,dr\,d\varphi$$

7, continued 2

We also have $u'_t(r,\varphi,0) = g(r,\varphi)$

which gives

$$g(r,\varphi) = \sum_{n=-\infty}^{\infty} \sum_{s=1}^{\infty} J_n\left(a_{ns}\frac{r}{r_0}\right)\left\{A_n \cos(n\varphi) + B_n \sin(n\varphi)\right\} d_{ns}\, \omega_{ns}$$

and

$$d_{ms}\, \omega'_{ms} \cdot \begin{Bmatrix} A_m \\ B_m \end{Bmatrix} \int_0^{r_0} \left[J_m\left(a_{mq}\frac{r}{r_0}\right)\right]^2 r\,dr \cdot \int_0^{2\pi} \begin{Bmatrix} \cos^2(m\varphi) \\ \sin^2(m\varphi) \end{Bmatrix} d\varphi =$$

$$= \int_0^{r_0}\int_0^{2\pi} J_m\left(a_{mq}\frac{r}{r_0}\right) \begin{Bmatrix} \cos(m\varphi) \\ \sin(m\varphi) \end{Bmatrix} g(r,\varphi)\, r\,dr\,d\varphi$$

8. The functions $u_l(x)$ and $u_m(x)$ are two different solutions to the differential equation $\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+(\lambda g(x)+\gamma)u(x)=0$ corresponding to different values, l and m, of λ. γ is a constant and $p(x)$ satisfies $p(a)=p(b)=0$. Show that with $g(x)$ as weight function $u_l(x)$ and $u_m(x)$ are orthogonal over the interval $[a,b]$. (2p)

8. We have $\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+(\lambda g(x)+\gamma)u(x)=0$

scale the equation for u_l by u_m and that for u_m by u_l

Integrate over the interval $[a,b]$ and take the difference

$$\text{I} \quad \int_a^b u_l \frac{d}{dx}\left[p(x)\frac{du_m}{dx}\right]dx = -\lambda_m\int_a^b u_l g u_m\,dx - \gamma\int_a^b u_l u_m\,dx$$

$$\text{II} \quad \int_a^b u_m \frac{d}{dx}\left[p(x)\frac{du_l}{dx}\right]dx = -\lambda_l\int_a^b u_m g u_l\,dx - \gamma\int_a^b u_m u_l\,dx$$

$$\text{I}-\text{II}: (\lambda_l-\lambda_m)\int_a^b u_m g(x) u_l\,dx = \int_a^b u_l\frac{d}{dx}\left[p(x)\frac{du_m}{dx}\right]dx - \int_a^b u_m\frac{d}{dx}\left[p(x)\frac{du_l}{dx}\right]dx$$

integrate by parts $= \left[u_l\, p(x)\frac{du_m}{dx}\right]_a^b - \int_a^b p(x)\frac{du_l}{dx}\frac{du_m}{dx}dx -$

$$- \left[u_m\, p(x)\frac{du_l}{dx}\right]_a^b + \int_a^b p(x)\frac{du_m}{dx}\frac{du_l}{dx}dx$$

$= 0$ since $p(a)=p(b)=0$

9. Use the Laplace transform technique to determine the oscillatory motion of a long wire which is initially at rest along the x-axis. Starting at time t=0 the end at x=0 is oscillated up and down so that $y(0,t) = 2\sin 3t$, $t > 0$. Find the displacement $y(x,t)$ at subsequent times with the initial and boundary conditions $y(0,t) = 2\sin 3t$, $y(x,0) = 0$, and $\frac{\partial y}{\partial t}\Big|_{t=0} = 0$. Take the Laplace transform of the wave equation and these conditions with respect to t to find the solution. (4p)

9. The wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

Initial conditions:

$y(0,t) = 2\sin 3t$, $t > 0$

$y(x,0) = 0$

$\frac{\partial y}{\partial t}\Big|_{t=0} = 0$

Let $Y(x,s) = \int_0^\infty e^{-st} y(x,t)\,dt$

Transform the equation

$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{\partial^2}{\partial x^2} L\{y\} = \frac{\partial^2 Y(x,s)}{\partial x^2}$$

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = s^2 Y - s\,y(x,0) - \frac{\partial y}{\partial t}\Big|_{t=0} = s^2 Y$$

Transform the initial condition at $x=0$

$$L\{2\sin 3t\} = 2\,\frac{3}{s^2+9} = \frac{6}{s^2+9}$$

The equation is now (regarding s as a constant)

$$\frac{d^2Y}{dx^2} = \frac{s^2}{v^2}Y \Rightarrow Y(x,s) = Ae^{-\frac{s}{v}x} + Be^{\frac{s}{v}x}$$

Set $B = 0$ (long wire)

$$Y(0,s) = A = \frac{6}{s^2+9}$$

$$\therefore Y(x,s) = \frac{6e^{-\frac{s}{v}x}}{s^2+9}$$

9, continued

The Bromwich integral

$$\mathcal{L}^{-1}\{Y(x,s)\} = \frac{6}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}e^{-\frac{s}{v}x}}{s^2+9}\,ds = 6\sum \text{enclosed residues}$$

There are poles at $s = \pm 3i$

For $x < vt$ we can close in the left half-plane and obtain

$$\mathcal{L}^{-1}\{Y(x,s)\} = 6\left\{\operatorname{Res}\{z=3i\} + \operatorname{Res}\{z=-3i\}\right\}$$

$$\operatorname{Res}\{z=3i\} = \left.\frac{\cancel{(z-3i)}\,e^{\frac{z}{v}(vt-x)}}{\cancel{(z-3i)}(z+3i)}\right|_{z=3i} = \frac{1}{6i}e^{3i(t-\frac{x}{v})}$$

$$\operatorname{Res}\{z=-3i\} = \left.\frac{\cancel{(z+3i)}\,e^{\frac{z}{v}(vt-x)}}{(z-3i)\cancel{(z+3i)}}\right|_{z=-3i} = -\frac{1}{6i}e^{-3i(t-\frac{x}{v})}$$

$\therefore$ for $x < vt$ we have $\mathcal{L}^{-1}\{Y(x,s)\} = \frac{1}{i}\left(e^{3i(t-\frac{x}{v})} - e^{-3i(t-\frac{x}{v})}\right) =$

$$= 2\sin 3\left(t - \frac{x}{v}\right)$$

For $x > vt$ we must close in the right half-plane excluding the poles giving $y(x,t) = 0$, $x > vt$