Written Examination for Mathematical Methods of Physics
2010.10.30 at 09:00-14:00

Allowed help: "Arfken and Weber" (or "Weber and Arfken"), "Physics Handbook", "Beta: Mathematics Handbook"

In order to get full credit:
1) Used formalisms should be clearly defined
2) All steps in your derivations that are based on references in the above books should be clearly given through reference to the relevant equations or tables.

1. Use calculus of residues to evaluate the integral $\int_0^{2\pi} \frac{d\vartheta}{5+4\cos\vartheta}$. (1p)

2. Prove that if two functions, f and g, are related by $g(x) = f(x-a)$ then their Fourier transforms, $\hat{f}$ and $\hat{g}$, are related by $\hat{g}(\omega) = e^{-ia\omega}\hat{f}(\omega)$ (2p)

3. Express the Bessel functions $J_2(x)$, $J_3(x)$ and $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$ (3p)

4. We consider the equation $2xy'' + (2-x)y' + \lambda y = 0$ on the interval $0 < x < \infty$. Here, $y \equiv y(x)$ and λ is a real parameter.

a) Make the following ansatz: $y(x) = x^k \sum_{n=0}^{\infty} a_n x^n$ and normalize it so that $a_0 = 1$.
Determine the value of k and find a recurrence relation for the coefficients a_n so that the differential equation becomes satisfied. (3p)

b) In the case $\lambda = 2$, determine all the coefficients a_n explicitly, write out the solution $y(x)$ and verify that it solves the equation. (2p)

c) By multiplying with a suitable factor, rewrite the differential equation on Sturm-Liouville form: $\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda w(x))y = 0$. Identify the weight function $w(x)$. (2p)

d) Assume that λ is an integer ≥ 0. Write $\lambda = n$, $n = 0,1,2\ldots$, and let $y_n \equiv y_n(x)$ denote the solution obtained in **a**), belonging to λ_n. Then you know that the set of functions $\{y_n(x)\}$ $n = 0,1,2\ldots$ forms a complete orthonormal system: $\int_0^{\infty} y_m(x) y_n(x) w(x) dx = \begin{cases} 0, & m \neq n \\ > 0, & m = n \end{cases}$

where for $m = n$ the result is non-zero but finite. Each function that satisfies $\int_0^\infty |f(x)|^2 w(x)dx < \infty$ can be expanded in a series as $f(x) = \sum_{n=0}^{\infty} A_n y_n(x)$. Derive a formula that gives the coefficients A_n. (3p)

5. The current $I(t)$ in an electric circuit with inductance L and resistance R is given by the equation $L\frac{dI}{dt} + RI = E(t)$ where $E(t)$ is the impressed electromotive force. (3p)

If $I(0) = 0$, use Laplace transforms to find $I(t)$ if:

a) $E(t) = E_0 u(t)$

b) $E(t) = E_0 \delta(t)$

c) $E(t) = E_0 \sin(\omega t)$

6. Find the Green's function associated with the boundary value problem

$$\frac{d}{dx}\left(x\frac{d\Psi}{dx}\right) - \frac{n^2}{x}\Psi = -f(x),\ 0 \le x \le 1,\ \Psi(0) \text{ finite},\ \Psi(1) = 0. \qquad (4p)$$

7. The wave equation for a circular membrane with radius a is given in polar coordinates (r, φ) by

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}$$ where $r \le a$, $0 \le \varphi \le 2\pi$, $t \ge 0$ and

$u \equiv u(r, \varphi, t)$ gives the bulge of the membrane. The membrane is fixed along $r = a$, which gives the boundary condition $u(a, \varphi, t) = 0$ for $0 \le \varphi \le 2\pi$ and $t \ge 0$.

a) Find the general solution to the above problem under the assumption that we have circular symmetry, *i.e.* that u is independent of φ so that it can be written $u \equiv u(r, t)$ (4p)

b) Find the solution obtained when $a = 2$, $c = 2$ and the initial conditions are

$$u(r,0) = 5J_0\left(\frac{\alpha_3 r}{2}\right)$$

$$\frac{\partial u}{\partial t}(r,0) = 4\alpha_7 J_0\left(\frac{\alpha_7 r}{2}\right) \qquad (3p)$$