Written Examination for Mathematical Methods of Physics
2010.01.04 at 09:00-15:00

Allowed help: "Arfken and Weber" (or "Weber and Arfken"), "Physics Handbook", "Beta: Mathematics Handbook"

In order to get full credit:
1) Used formalisms should be clearly defined
2) All steps in your derivations that are based on references in the above books should be clearly given.

1. Evaluate the integral $\int_0^{2\pi} \frac{d\vartheta}{13+5\sin\vartheta}$ (2p)

2. Take the Fourier transform of the function $f(x)=\begin{cases}1, & \pi/2<|x|<\pi \\ 0, & otherwise\end{cases}$ and write $f(x)$ in terms of its Fourier expansion. (2p)

3. Use normalized Legendre polynomials $p_l(x)$, $p_l(x)=\sqrt{\frac{2l+1}{2}}P_l(x)$, to expand the function $f(x)=x^4$ on the interval -1<x<1. (3p)

4. Solve the differential equation $y''+2y'+2y=10e^x+6e^{-x}\cos x$. Note that a particular solution can be obtained according to problem (A&W 9.6.25) as (3p)

$$y_p(x)=y_2(x)\int^x \frac{y_1(s)F(s)ds}{W\{y_1(s),y_2(s)\}}-y_1(x)\int^x \frac{y_2(s)F(s)ds}{W\{y_1(s),y_2(s)\}}$$

with $W\{y_1(s),y_2(s)\}$ the Wronskian of the homogeneous solutions $y_1(s)$ and $y_2(s)$ and $F(s)$ the inhomogeneous right hand side of the equation

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=F(x).$$

5. Find the Green's function for the problem $y''+y=f(x)$ with boundary conditions $y(0)=y(\pi/2)=0$. Write the solution $y(x)$ in terms of this Green's function. (3p)

6. A flat circular plate of radius a has an initial temperature distribution $u(r,\theta,t=0)=100r\sin\theta$. From time $t=0$ on, the circumference of the plate is held at 0°. Find the time-dependent temperature distribution $u(r,\theta,t)$. Hint: Separate variables in the heat-flow equation in polar coordinates. (4p)

7. Use the series definition of the Bessel functions, $J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$ to explicitly verify that: (3p)

(a) $J_1(x) + J_3(x) = (4/x)J_2(x)$

(b) $\frac{d}{dx}(xJ_1(x)) = xJ_0(x)$

8. Use the Frobenius technique to solve the equation $x^2y'' + 4xy' + (x^2 + 2)y = 0$. (4p)

9. Show that the inverse Laplace transform $L^{-1}\{(s^2 + a^2)^{-2}\} = \frac{1}{2a^3}\sin at - \frac{1}{2a^2}t\cos at$ (3p)

10. Use the Laplace transform technique and the convolution integral to solve the equation $y'' + 3y' + 2y = e^{-t}$, where $y(0) = y'(0) = 0$ and $y(t) = 0$ for $t < 0$. (3p)