Written Examination for Mathematical Methods of Physics
2009.10.24 at 09:00-15:00

Allowed help: "Arfken and Weber" (or "Weber and Arfken"), "Physics Handbook", "Beta: Mathematics Handbook"

In order to get full credit:
1) Used formalisms should be clearly defined
2) <u>All steps in your derivations</u> that are based on references in the above books should be clearly given.

1. Use calculus of residues to evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos(kx)dx}{x^2+a^2}$, $k > 0$. (2p)

2. Take the Fourier cosine transform of the function $f(x) = \begin{cases} 1-|x|, & -1 < x < 1 \\ 0, & |x| > 1 \end{cases}$ and write $f(x)$ in terms of its Fourier expansion. (2p)

3. Expand the polynomial $G(\vartheta) = 2\cos^2\vartheta\sin^2\vartheta - 3\cos^2\vartheta + 2$ in a series of Legendre polynomials. (3p)

4. Solve the differential equation $y'' + y' - 2y = e^x + x$. Note that a particular solution can be obtained according to problem (A&W 9.6.25) as (3p)

$$y_p(x) = y_2(x)\int^x \frac{y_1(s)F(s)ds}{W\{y_1(s), y_2(s)\}} - y_1(x)\int^x \frac{y_2(s)F(s)ds}{W\{y_1(s), y_2(s)\}}$$

with $W\{y_1(s), y_2(s)\}$ the Wronskian of the homogeneous solutions $y_1(s)$ and $y_2(s)$ and $F(s)$ the inhomogeneous right hand side of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x).$$

5. ~~Use the Laplace transform technique to solve the differential equation~~ $y'' + \omega^2 y = \delta(t - t_0)$ for $t > t_0$ and initial conditions $y(0) = y'(0) = 0$. (3p)

6. The walls of an "infinite slab" (*i.e.* large enough so that edge effects can be neglected) of length l are initially held at constant temperature $T = 0$ for the side at $x = 0$ and $T = 100°$ for that at $x = l$. Heat is only flowing in the x-direction so that the problem can be considered as one-dimensional.
(a) Find the steady-state temperature distribution for this situation. (1p)

(b) At time $t = 0$ the side at $x = l$ is also connected to a heat bath with temperature $T = 0$. Find the subsequent temperature distribution as function of x and t. (3p)

7. Use the series definition of the Legendre functions, $P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$ to explicitly verify the recurrence relation $P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x)$ (3p)
(Hint: For n even, *i.e.* $n = 2p$, $[(n+1)/2] = [n/2] = p$.
For n odd, *i.e.* $n = 2p+1$, $[(n+1)/2] = p+1 \neq [n/2] = p$, but the extra terms with $k = p + 1$ vanish due to the factorial $(-1)! \to \infty$ in the denominators).

8. Use the Frobenius technique to find the solution with lowest power zero (*i.e.* $p = 0$ in the ansatz $y(x) = \sum_{k=0}^{\infty} a_k x^{k+p}$) to the equation $2xy'' + y' + 2y = 0$ (3p)
(Hint: The double factorial $(2n-1)!!$ can be written $(2n-1)!! = \frac{(2n)!}{2^n n!}$)

9. Evaluate the integral $\int_0^{\infty} \frac{dx}{1+x^3}$. Be careful to specify the contour used! (3p)

10. Use the Laplace transform technique and the defining equation of the Green's function to find the Green's function for the equation $y'' + 2y' + y = f(t)$, where $y(0) = y'(0) = 0$ and $y(t) = 0$ for $t < 0$. (4p)