Written Examination for Mathematical Methods of Physics
2017.10.27 at 08:00-13:00

Allowed help: "Arfken, Weber and Harris" (or "Arfken and Weber"), "Physics Handbook", "Beta: Mathematics Handbook" and the handed-out lecture notes from the course.

In order to get full credit:
1) Used formalisms should be clearly defined
2) <u>All steps in your derivations</u> that are based on references in the above books should be clearly given through reference to the relevant equations or tables.

Note that problems 6 - 9 are on the back.

1. Use calculus of residues to evaluate the integral $\int_0^\infty \frac{\sin(x)dx}{x(x^2+a^2)}$. Specify the contour used!

(3p)

2. Use a method based on contour integration to evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$ (4p)

3a. Use the Frobenius method to find the odd and even solutions to the equation (derived from the quantum mechanical harmonic oscillator):

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (E-1)y = 0$$

Determine values of the energy E such that the series terminate resulting in polynomials of finite order. (3p)

b. Write down explicitly the polynomials corresponding to the three lowest energies as obtained from your expansion and give their energies (the units here are arbitrary). (1p)

4. Use the Cauchy integral formula to construct a function $f(z)$ satisfying the properties:

(a) $f(z)$ is analytic except for a simple pole of residue *R* at *z*=*a* and a branch cut $(0,\infty)$ at which the function has a discontinuity $f(x+i\varepsilon) - f(x-i\varepsilon) = 2i\pi g(x),\ x \geq 0$.

(b) $|f(z)| \to 0$ as $|z| \to \infty$, and $|zf(z)| \to 0$ as $|z| \to 0$.

Be careful with the specification of contours used and express the function in terms of *R*, *a*, and *g*(*x*). (3p)

5. Three radioactive nuclei decay successively in series, so that the numbers $N_i(t)$ of the three types obey the equations

$$\frac{dN_1}{dt} = -\lambda_1 N_1$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$$

$$\frac{dN_3}{dt} = \lambda_2 N_2 - \lambda_3 N_3$$

(3p)

If initially $N_1 = N,\ N_2 = 0,\ N_3 = n$, find $N_3(t)$ by using Laplace transforms.

6. $f_n(x)$ are polynomials of order n (n= 0, 1, 2,…) which are mutually orthogonal on the range 0 to ∞, with weight function exp(-x); that is, $\int_0^\infty e^{-x} f_n(x) f_m(x) dx = 0$ if $m \neq n$. Find $g(x)$ such that the $f_n(x)$ as defined above satisfy the equation:

$$x\frac{d^2 f_n}{dx^2} + g(x)\frac{df_n}{dx} + \lambda_n f_n = 0 \qquad \text{(2p)}$$

7. For $m = 0$ the Bessel equation is $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0$. Assuming that you have found a first solution $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{36\cdot 64} + \cdots$ show that a second solution exists of the form $J_0(x)ln(x) + Ax^2 + Bx^4 + Cx^6 + \cdots$ and find the first three coefficients A, B, C. (3p)

8. An oscillator initially at rest with $X(0) = X'(0) = 0$ is subjected to a driving force *f*(*t*) where $f(t) = \begin{cases} 0 & t < 0 \\ \gamma e^{-at} & t \geq 0 \end{cases}$ and *a* > 0.

The equation describing the subsequent motion can be written

$$\frac{d^2 X(t)}{dt^2} + \omega_0^2 X(t) = f(t).$$

Use the Fourier transform to find the retarded Green function $G_r(t)$ and use this Green's function to construct the solution for *X*(*t*), *t* > 0. (The retarded Green's function obeys causality, *i.e*. the response comes after the perturbation $f(t)$). Verify that your solution satisfies the initial conditions! (4p)

9. A conducting sphere of radius *a* is divided into two electrostatically separated hemispheres by a thin insulating barrier at its equator. The top hemisphere is maintained at a potential V_0, and the bottom hemisphere at $-V_0$. Show that the potential *exterior* to the two hemispheres is

$$V(r,\theta) = V_0 \sum_{s=0}^{\infty} (-1)^s (4s+3) \frac{(2s-1)!!}{(2s+2)!!} \left(\frac{a}{r}\right)^{2s+2} P_{2s+1}(cos\theta)$$

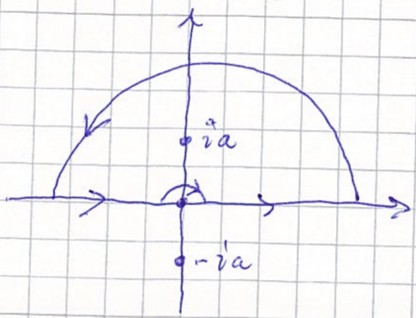
where $P_{2s+1}(cos\theta)$ is the Legendre polynomial of order *2s+1* (4p)

1. Use calculus of residues to evaluate the integral $\int_0^\infty \frac{\sin(x)dx}{x(x^2+a^2)}$. Specify the contour used!

(3p)

$$\int_0^\infty \frac{\sin x}{x(x^2+a^2)}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin x}{x(x^2+a^2)}dx = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+a^2)}dx$$

The denominator goes as $\frac{1}{x^3}$ for large x and the argument of the exponential is positive. Use Jordan's lemma to close in the UHP. Note the pole at $x=0$ on the path of integration.



Poles at $z=0, \pm ia$

The pole at $z=ia$ is included. Exclude the pole at $z=0$ with a half circle

$$\mathrm{Res}\{z=ia\} = \left.\frac{(z-ia)e^{iz}}{z(z+ia)(z-ia)}\right|_{z=ia} = -\frac{e^{-a}}{2a^2}$$

$$\mathrm{Res}\{z=0\} = \left.\frac{\cancel{z}e^{iz}}{\cancel{z}(z^2+a^2)}\right|_{z=0} = \frac{1}{a^2}$$

$$\therefore \int_0^\infty \frac{\sin x}{x(x^2+a^2)}dx = \frac{1}{2}\,\mathrm{Im}\left\{\pi i\cdot\frac{1}{a^2} - 2\pi i\frac{e^{-a}}{2a^2}\right\} = \frac{\pi}{2a^2}\left(1-e^{-a}\right)$$

2. Use a method based on contour integration to evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$ (4p)

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{1}{16} \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^4} = \frac{1}{32} \sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^4}$$

Use that $\pi \tan \pi z = \pi \frac{\sin \pi z}{\cos \pi z}$ has poles at $z = \pm n + \frac{1}{2}$

with residues $a_{-1} = \left. \frac{\pi \sin \pi z}{-\pi \sin \pi z} \right|_{z=n+\frac{1}{2}} = -1$

Thus, we need the residue of $\frac{\pi \tan \pi z}{z^4}$ at the fourth order pole $z=0$

General expression: $a_{-1} = \frac{1}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} (x-x_0)^n f(x)$

$$\frac{d}{dz} \pi \tan \pi z = \pi \frac{d}{dz} \frac{\sin \pi x}{\cos \pi x} = \pi^2 \left\{ \frac{\cos \pi z}{\cos \pi z} + \frac{\sin^2 \pi z}{\cos^2 \pi z} \right\} = \pi^2 \left\{ 1 + \frac{\sin^2 \pi z}{\cos^2 \pi z} \right\}$$

$$\frac{d^2}{dz^2} \pi \tan \pi z = \pi^3 \left\{ \frac{2 \sin \pi z \cdot \cos \pi z}{\cos^2 \pi z} + \frac{2 \sin^3 \pi z}{\cos^3 \pi z} \right\} = \pi^3 \left\{ 2 \frac{\sin \pi z}{\cos \pi z} + 2 \frac{\sin^3 \pi z}{\cos^3 \pi z} \right\}$$

$$\frac{d^3}{dz^3} \pi \tan \pi z = \pi^4 \left\{ 2 + 2 \frac{\sin^2 \pi z}{\cos^2 \pi z} + \frac{6 \sin^2 \pi z \cos \pi z}{\cos^3 \pi z} + 6 \frac{\sin^4 \pi z}{\cos^4 \pi z} \right\}$$

Evaluate at $z=0$ $\Rightarrow \left. \frac{d^3}{dz^3} \frac{(z-0)^4 \pi \tan \pi z}{z^4} \right|_{z=0} = 2\pi^4$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{1}{32} \cdot \frac{1}{3!} \cdot 2\pi^4 = \frac{\pi^4}{96}$$

3. Use the Frobenius method to find two independent solutions to the equation:

$$3x\frac{d^2y}{dx^2} + (3x+1)\frac{dy}{dx} + y = 0$$

One sums to an elementary function. For the other give the four first terms of the sum (4p)

Make the ansatz $y(x) = \sum_{k=0}^{\infty} a_k x^{k+s}$ with $a_0 \neq 0$

$y'(x) = \sum_{k=0}^{\infty} a_k (k+s) x^{k+s-1}$; $y''(x) = \sum_{k=0}^{\infty} a_k (k+s)(k+s-1) x^{k+s-2}$

The equation $3\sum_{k=0}^{\infty} a_k (k+s)(k+s-1) x^{k+s-1} + 3\sum_{k=0}^{\infty} a_k (k+s) x^{k+s} +$

$+ \sum_{k=0}^{\infty} a_k (k+s) x^{k+s-1} + \sum_{k=0}^{\infty} a_k x^{k+s} = 0$

Lowest power x^{s-1}: $3a_0 s(s-1) + a_0 s = 0 \to 3s^2 - 2s = 0$

$\therefore$ $s = 0,\ 2/3$

Recurrence relation: $a_{k+1}\left[3(k+s)(k+s+1) + (k+s+1)\right] + a_k\left[3(k+s)+1\right] = 0$

$$a_{k+1} = -\frac{3(k+s)+1}{\left[3(k+s)+1\right](k+s+1)} a_k = -\frac{1}{k+s+1} a_k$$

$\underline{s = 0}$:

$a_{k+1} = -\frac{1}{k+1} a_k$

$a_1 = -a_0$

$a_2 = -\frac{1}{2}a_1 = \frac{1}{2}a_0$

$a_3 = -\frac{1}{3}a_2 = -\frac{1}{3\cdot 2}a_0$

$a_n = \frac{(-1)^n}{n!} a_0$

$\therefore$ $\underline{\underline{y_1(x) = a_0 e^{-x}}}$

$s = \frac{2}{3}$:

$a_{k+1} = -\frac{1}{k+\frac{2}{3}+1} a_k = -\frac{3}{3k+5} a_k$

$a'_1 = -\frac{3}{5} a'_0$

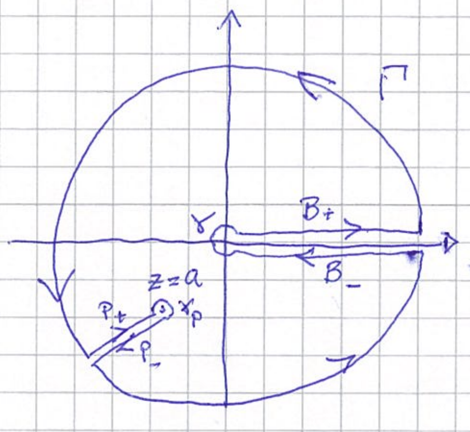
$a'_2 = -\frac{3}{8} a'_1 = \frac{3^2}{5\cdot 8} a'_0$

$a'_3 = -\frac{3}{11} a'_2 = -\frac{3^3}{5\cdot 8\cdot 11} a'_0$

$$\underline{\underline{y_2(x) = a'_0 x^{2/3}\left\{1 - \frac{3x}{5} + \frac{(3x)^2}{5\cdot 8} - \frac{(3x)^3}{5\cdot 8\cdot 11} + \dots\right\}}}$$

4. Use the Cauchy integral formula to construct a function $f(z)$ satisfying the properties:
(a) $f(z)$ is analytic except for a simple pole of residue R at $z=a$ and a branch cut $(0,\infty)$ at which the function has a discontinuity $f(x+i\varepsilon)-f(x-i\varepsilon)=2i\pi g(x)$, $x \geq 0$.
(b) $|f(z)| \to 0$ as $|z| \to \infty$, and $|zf(z)| \to 0$ as $|z| \to 0$.
Be careful with the specification of contours used and express the function in terms of R, a, and $g(x)$. (3p)

We have $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z'-z} dz' = \frac{1}{2\pi i} \left\{ \int_\Gamma + \int_{P_+} + \oint_{\gamma_p} + \int_{P_-} + \int_{B_-} + \int_\gamma + \int_{B_+} \right\}$



The integral along contour Γ gives zero since $|f(z)| \to 0$ as $|z| \to \infty$

The integral along γ gives zero since $|zf(z)| \to 0$ as $|z| \to 0$

The integrals along P_+ and P_- cancel

We thus have $f(z) = \frac{1}{2\pi i} \oint_{\gamma_p} \frac{f(z')dz'}{z'-z} + \frac{1}{2\pi i} \int_{B_-} \frac{f(z')dz'}{z'-z} + \frac{1}{2\pi i} \int_{B_+} \frac{f(z')dz'}{z'-z}$

The simple pole at $z=a$: $\frac{1}{2\pi i} \oint_{\gamma_p} \frac{f(z')}{z'-z} dz' = \left| \begin{array}{l} z' = a + \delta e^{i\vartheta} \\ dz' = i\delta e^{i\vartheta} d\vartheta \\ z'-z = a - z + \delta e^{i\vartheta} \end{array} \right| =$

$= \lim_{\delta \to 0} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(a+\delta e^{i\vartheta})\delta e^{i\vartheta} d\vartheta}{a-z+\delta e^{i\vartheta}} = -\frac{1}{2\pi} \cdot \frac{1}{a-z} \lim_{\delta \to 0} \int_0^{2\pi} \frac{f(a+\delta e^{i\vartheta})\delta e^{i\vartheta} d\vartheta}{1 + \frac{\delta e^{i\vartheta}}{a-z}}$

Use the Laurent expansion of $f(z)$ around a (simple pole)

$f(z) = \frac{R}{z-a} + \sum_{n=0}^{\infty} a_n (z-a)^n \to f(a+\delta e^{i\vartheta}) = \frac{R}{\delta e^{i\vartheta}} + \sum_{n=0}^{\infty} a_n \delta^n e^{in\vartheta}$

$\lim_{\delta \to 0} \int_0^{2\pi} \frac{f(a+\delta e^{i\vartheta})\delta e^{i\vartheta} d\vartheta}{1 + \frac{\delta e^{i\vartheta}}{a-z}} = \lim_{\delta \to 0} \int_0^{2\pi} \frac{R\, d\vartheta}{1 + \frac{\delta e^{i\vartheta}}{a-z}} + \lim_{\delta \to 0} \sum_{n=0}^{\infty} a_n \delta^{n+1} \int_0^{2\pi} \frac{e^{i(n+1)\vartheta} d\vartheta}{1 + \frac{\delta e^{i\vartheta}}{a-z}}$

$\Rightarrow f(z) = -\frac{1}{2\pi} \frac{2\pi R}{a-z} + \frac{1}{2\pi i} \int_0^{\infty} \frac{f(x'+i\varepsilon) - f(x'-i\varepsilon)}{x'-z} dx'$

$\Rightarrow f(z) = \frac{R}{z-a} + \int_0^{\infty} \frac{g(x')}{x'-z} dx'$

5. Three radioactive nuclei decay successively in series, so that the numbers $N_i(t)$ of the three types obey the equations

$$\frac{dN_1}{dt} = -\lambda_1 N_1$$
$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \qquad \text{(3p)}$$
$$\frac{dN_3}{dt} = \lambda_2 N_2 - \lambda_3 N_3$$

If initially $N_1 = N,\ N_2 = 0,\ N_3 = n$, find $N_3(t)$ by using Laplace transforms. The transforms will all be similar. Do one inversion explicitly using the Bromwich integral and the remaining by analogy.

The Laplace transform of a derivative

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_0^\infty e^{-st}\frac{df}{dt}\,dt = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt = s\tilde{f}(s) - f(0)$$

Transform each equation: $s\tilde{N}_1(s) - N = -\lambda_1 \tilde{N}_1(s)$

$$s\tilde{N}_2(s) = \lambda_1 \tilde{N}_1(s) - \lambda_2 \tilde{N}_2(s)$$
$$s\tilde{N}_3(s) - n = \lambda_2 \tilde{N}_2(s) - \lambda_3 \tilde{N}_3(s)$$

$$\tilde{N}_1(s+\lambda_1) = N \quad \rightarrow \tilde{N}_1 = \frac{N}{s+\lambda_1}$$

$$\tilde{N}_2(s+\lambda_2) = \lambda_1 \tilde{N}_1 = \frac{\lambda_1}{s+\lambda_1}N \longrightarrow \tilde{N}_2 = \frac{\lambda_1}{(s+\lambda_1)(s+\lambda_2)}N$$

$$\tilde{N}_3(s+\lambda_3) = n + \lambda_2\tilde{N}_2 = n + \frac{\lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)}N \rightarrow \tilde{N}_3 = \frac{n}{s+\lambda_3} + \frac{\lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)}N$$

Invert:

$$N_1(t) = \frac{N}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}}{s+\lambda_1}\,ds = N\cdot \mathrm{Res}\{s=-\lambda_1\} = Ne^{-\lambda_1 t}$$

$$N_2(t) = \frac{\lambda_1 N}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}\,ds}{(s+\lambda_1)(s+\lambda_2)} = \lambda_1 N\cdot\left\{\mathrm{Res}[s=-\lambda_1] + \mathrm{Res}[s=-\lambda_2]\right\} =$$

$$= \lambda_1\cdot N\left\{\frac{e^{-\lambda_1 t}}{\lambda_2-\lambda_1} + \frac{e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right\} = \frac{\lambda_1\cdot N}{\lambda_2-\lambda_1}\left\{e^{-\lambda_1 t} - e^{-\lambda_2 t}\right\}$$

$$N_3(t) = ne^{-\lambda_3 t} + \lambda_1\lambda_2 N\left\{\frac{e^{-\lambda_1 t}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} + \frac{e^{-\lambda_3 t}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right\}$$

(5.2)

Check the solution:
$$\frac{dN_1}{dt} = -\lambda_1 N e^{-\lambda_1 t} = -\lambda_1 N_1$$

$$N_1(0) = N$$

$$\frac{dN_2}{dt} = \frac{\lambda_1 \cdot N}{\lambda_2 - \lambda_1}\left\{-\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}\right\} =$$

$$= \frac{\lambda_1 N}{\lambda_2 - \lambda_1}\left\{\lambda_2 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}\right\} =$$

$$= \frac{\lambda_1 N}{\lambda_2 - \lambda_1}(\lambda_2 - \lambda_1)e^{-\lambda_1 t} - \frac{\lambda_2 \lambda_1 N}{\lambda_2 - \lambda_1}\left\{e^{-\lambda_1 t} - e^{-\lambda_2 t}\right\} = \lambda_1 N_1 - \lambda_2 N_2$$

$$N_2(0) = 0$$

$$\frac{dN_3}{dt} = -\lambda_3 n e^{-\lambda_3 t} + \lambda_1 \lambda_2 N \left\{\frac{+\lambda_1 e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{\lambda_2 e^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} + \lambda_3 \frac{e^{-\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\right\}$$

$$= -\lambda_3 n e^{-\lambda_3 t} - \frac{\lambda_1 \lambda_2 N}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\left\{\lambda_1(\lambda_3 - \lambda_2)e^{-\lambda_1 t} - \lambda_2(\lambda_3 - \lambda_1)e^{-\lambda_2 t} + \lambda_3(\lambda_2 - \lambda_1)e^{-\lambda_3 t}\right\}$$

$$= -\lambda_3 n e^{-\lambda_3 t} - \frac{\lambda_1 \lambda_2 N}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\left\{-\lambda_3(\lambda_3 - \lambda_2)e^{-\lambda_1 t} + \lambda_3(\lambda_3 - \lambda_2)e^{-\lambda_1 t} + \right.$$

$$\left. + \lambda_1(\lambda_3 - \lambda_2)e^{-\lambda_1 t} + \lambda_3(\lambda_3 - \lambda_1)e^{-\lambda_2 t} - \lambda_3(\lambda_3 - \lambda_1)e^{-\lambda_2 t} - \lambda_2(\lambda_3 - \lambda_1)e^{-\lambda_2 t} + \lambda_3(\lambda_2 - \lambda_1)e^{-\lambda_3 t}\right\}$$

$$= -\lambda_3 n e^{-\lambda_3 t} + \frac{\lambda_1 \lambda_2 N}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\left[\left\{-(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)e^{-\lambda_1 t} + (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)e^{-\lambda_2 t}\right\}\right.$$

$$\left. + \lambda_3(\lambda_3 - \lambda_2)e^{-\lambda_1 t} - \lambda_3(\lambda_3 - \lambda_1)e^{-\lambda_2 t} + \lambda_3(\lambda_2 - \lambda_1)e^{-\lambda_3 t}\right] =$$

$$= -\lambda_3 n e^{-\lambda_3 t} + \lambda_2 N_2 - \lambda_3 \frac{\lambda_1 \lambda_2 N}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\left\{(\lambda_3 - \lambda_2)e^{-\lambda_1 t} - (\lambda_3 - \lambda_1)e^{-\lambda_2 t} + (\lambda_2 - \lambda_1)e^{-\lambda_3 t}\right\}$$

$$\cancel{= -\lambda_3}$$

$$= \lambda_2 N_2 - \lambda_3 \left\{n e^{-\lambda_3 t} + \lambda_1 \lambda_2 N \left\{\frac{e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} + \frac{e^{-\lambda_3 t}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\right\}\right\}$$

$$= \lambda_2 N_2 - \lambda_3 N_3$$

$$N_3(0) = n$$

6. $f_n(x)$ are polynomials of order n (n= 0, 1, 2,…) which are mutually orthogonal on the range 0 to ∞, with weight function exp(-x); that is, $\int_0^\infty e^{-x} f_n(x) f_m(x)dx = 0$ if $m \neq n$. Find $g(x)$ such that the $f_n(x)$ as defined above satisfy the equation:

$$x\frac{d^2 f_n}{dx^2} + g(x)\frac{df_n}{dx} + \lambda_n f_n = 0 \qquad (2p)$$

Use the weight function to put the equation on self-adjoint form.

Then $\frac{d}{dx}\left[xe^{-x}\frac{df_n}{dx}\right] = xe^{-x}\frac{d^2 f_n}{dx^2} + g(x)e^{-x}\frac{df_n}{dx}$

The left hand side expands to $(e^{-x} - xe^{-x})\frac{df_n}{dx} + xe^{-x}\frac{d^2 f_n}{dx^2} =$

$= e^{-x}(1-x)\frac{df_n}{dx} + xe^{-x}\frac{d^2 f_n}{dx^2}$

Identify $g(x)e^{-x} \equiv (1-x)e^{-x}$, i.e. $g(x) = 1-x$

Alternative: From the expression for the weight function

$$w(x) = \frac{1}{p(x)}\exp\left[\int^x \frac{q(x')}{p(x')}dx'\right] \text{ where } p(x) = x \text{ and } q(x) = g(x)$$

$$e^{-x} = \frac{1}{x}\exp\left[\int^x \frac{g(x')}{x'}dx'\right] \Leftrightarrow xe^{-x} = \exp\left[\int^x \frac{g(x')dx'}{x'}\right]$$

Take the logarithm: $\ln x - x = \int^x \frac{g(x')dx'}{x'}$

Differentiate: $\frac{1}{x} - 1 = \frac{g(x)}{x} \Rightarrow g(x) = 1-x$

7. For $m = 0$ the Bessel equation is $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 0$. Assuming that you have found a first solution $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{36\cdot 64} + \cdots$ show that a second solution exists of the form $J_0(x)ln(x) + Ax^2 + Bx^4 + Cx^6 + \cdots$ and find the first three coefficients A, B, C.

(3p)

Assume a solution of the given form and insert in the equation to determine A, B, C such that the corresponding powers vanish

$$\frac{d}{dx}\left[J_0(x)\ln(x) + Ax^2 + Bx^4 + Cx^6 + ..\right] = J_0'\ln x + \frac{1}{x}J_0 + 2Ax + 4Bx^3 + 6Cx^5$$

$$\frac{d^2}{dx^2}\left[\quad\right] = J_0''\ln x + \frac{1}{x}J_0' + \frac{1}{x}J_0' - \frac{1}{x^2}J_0 + 2A + 12Bx^2 + 30Cx^4 + \ldots$$

Insert in the equation and collect terms (to 6th order)

$$x^2y'' + xy' + x^2y = x^2J_0''\ln x + 2xJ_0' - \cancel{J_0} + 2Ax^2 + 12Bx^4 + 30Cx^6 +$$

$$+ xJ_0'\ln x + \cancel{J_0} + 2Ax^2 + 4Bx^4 + 6Cx^6 + x^2J_0\ln x + Ax^4 + Bx^6 + Cx^8 =$$

$$= \ln x\underbrace{\left\{x^2J_0'' + xJ_0' + x^2J_0\right\}}_{=0} + 2xJ_0' + x^2(2A + 2A) + x^4(12B + 4B + A) +$$

$$+ x^6(B + 6C + 30C) = 2xJ_0' + 4Ax^2 + (A + 16B)x^4 + (B + 36C)x^6;$$

$$2xJ_0' = 2x\frac{d}{dx}\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{36\cdot 64} + ..\right) = 2x\left(-\frac{x}{2} + \frac{x^3}{16} - \frac{x^5}{6\cdot 64} + ..\right) =$$

$$= -x^2 + \frac{x^4}{8} - \frac{x^6}{3\cdot 64} + \ldots$$

$$\Rightarrow x^2y'' + xy' + x^2y = -x^2 + \frac{x^4}{8} - \frac{x^6}{3\cdot 64} + 2Ax^2 + (A + 16B)x^4 + (B + 36C)x^6 =$$

$$= x^2(4A - 1) + \left(A + 16B + \frac{1}{8}\right)x^4 + \left(B + 36C - \frac{1}{3\cdot 64}\right)x^6$$

x^2: $4A = 1 \Rightarrow A = \frac{1}{4}$

x^4: $\frac{1}{4} + 16B + \frac{1}{8} = 0 \to 16B = -\frac{3}{8} \to B = -\frac{3}{8\cdot 16} = -\frac{3}{128}$

x^6: $-\frac{3}{128} + 36C - \frac{1}{3\cdot 64} = 0 \to 36C = \frac{3}{2\cdot 64} + \frac{1}{3\cdot 64} = \frac{11}{6\cdot 64} \to C = \frac{11}{6\cdot 64\cdot 36} = \frac{11}{27\cdot 512}$

Alternative: Follow the book example 7.6.4 and construct the second solution

$$y_2(x) = J_0 \int^x \frac{\exp\left[-\int^{x_2} x_1^{-1} dx_1\right]}{\left[1 - \frac{x_2^2}{4} + \frac{x_2^4}{64} - \frac{x_2^6}{36\cdot 64} + \ldots\right]^2} dx_2$$

The integral $\int^{x_2} \frac{dx_1}{x_1} = \ln x_2$ so $\exp\left[-\int^{x_2} \frac{dx_1}{x_1}\right] = \frac{1}{x_2}$

$$y_2(x) = J_0 \int^x \frac{dx_2}{x_2\left[1 - \frac{x_2^2}{4} + \frac{x_2^4}{64} - \frac{x_2^6}{36\cdot 64} + -\right]^2}$$

Do a binomial expansion of the denominator retaining powers up to x^6

$$\left(1 + \left(-\frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{36\cdot 64}\right)\right)^{-2} = 1 - 2\left(-\frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{36\cdot 64}\right) + 3\left(-\frac{x^2}{4} + \frac{x^4}{64}\right)^2 - 4\left(\frac{x^2}{4}\right)^3 + \ldots$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{32} + \frac{x^6}{32\cdot 36} + 3\frac{x^4}{16} - 3\frac{x^6}{128} + \frac{x^6}{16} =$$

$$= 1 + \frac{x^2}{2} + \frac{5}{32}x^4 + x^6\left(\frac{1}{9\cdot 128} + \frac{5}{128}\right) = 1 + \frac{x^2}{2} + \frac{5}{32}x^4 + \frac{23}{9\cdot 64}x^6$$

$$y_2(x) = J_0(x)\int^x \left(\frac{1}{x_2} + \frac{1}{2}x_2 + \frac{5}{32}x_2^3 + \frac{23}{9\cdot 64}x^5\right) dx_2 =$$

$$= J_0(x)\left\{\ln x + \frac{1}{4}x^2 + \frac{5}{128}x^4 + \frac{23}{27\cdot 128}x^6\right\} =$$

$$= J_0(x)\ln x + \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{36\cdot 64}\right)\left(\frac{1}{4}x^2 + \frac{5}{128}x^4 + \frac{23}{27\cdot 128}x^6\right) =$$

$$= J_0(x)\ln x + \frac{1}{4}x^2 + \frac{5}{128}x^4 + \frac{23}{27\cdot 128}x^6 - \frac{1}{16}x^4 - \frac{5}{4\cdot 128}x^6 + \frac{1}{4\cdot 64}x^6 =$$

$$= J_0(x)\ln x + \frac{1}{4}x^2 + x^4\left(\frac{5}{128} - \frac{1}{16}\right) + x^6\left(\frac{23}{27\cdot 128} - \frac{5}{4\cdot 128} + \frac{1}{4\cdot 64}\right) =$$

$$= J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + x^6\left(\frac{4\cdot 23 - 5\cdot 27 + 2\cdot 27}{4\cdot 27\cdot 128}\right) =$$

$$= J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{27\cdot 512}x^6$$

8. An oscillator initially at rest with $X(0) = X'(0) = 0$ is subjected to a driving force $f(t)$ where $f(t) = \begin{cases} 0 & t < 0 \\ \gamma e^{-at} & t \geq 0 \end{cases}$ and $a > 0$.

The equation describing the subsequent motion can be written

$$\frac{d^2X(t)}{dt^2} + \omega_0^2 X(t) = f(t).$$

Use the Fourier transform to find the retarded Green function $G_r(t)$ and use this Green's function to construct the solution for $X(t)$, $t > 0$. (The retarded Green's function obeys causality, *i.e.* the response comes after the perturbation $f(t)$). Verify that your solution satisfies the initial conditions! (4p)

The Fourier transform of the defining equation for $G(t,t')$

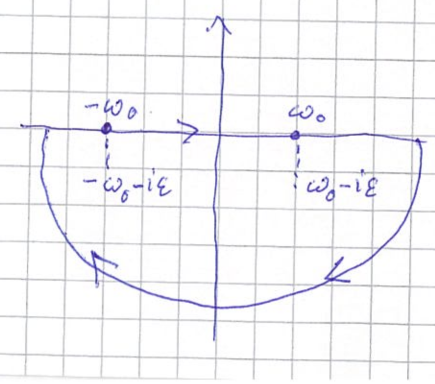
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{G'' + \omega_0^2 G\} e^{i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t-t') e^{i\omega t} dt$$

This gives $(-\omega^2 + \omega_0^2) g(\omega) = \frac{1}{\sqrt{2\pi}} e^{i\omega t'}$

and $$g(\omega) = \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega t'}}{\omega_0^2 - \omega^2}$$

Transform back: $$G(t,t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t')}}{\omega_0^2 - \omega^2} d\omega$$

We have poles $\omega = \pm\omega_0$ on the path of integration. The Green's function should obey the initial conditions $G(0) = G'(0) = 0$ and causality, i.e. for $t < t'$ we should have $G(t,t') = 0$. For $t < t'$ we have $-(t-t') > 0$ so we will close in the UHP (using Jordan's lemma). To obtain $G(t,t') = 0$ for $t < t'$ we move the poles into the lower half-plane using $\pm\omega_0 \to \pm\omega_0 - i\varepsilon$ and let $\varepsilon \to 0$ after the integration. No poles are then contained and $G(t,t') = 0$, $t < t'$.



For $t > t'$ we close in the LHP and use Jordan's lemma remembering that we go clockwise

$$\frac{1}{2\pi} \oint \frac{e^{-i\omega(t-t')}}{(\omega_0 - i\varepsilon)^2 - \omega^2} d\omega = -i \sum \{\text{enclosed residues}\}.$$

The simple poles at $\pm\omega_0 - i\varepsilon$ are enclosed

The residue at $\omega = \omega_0 - i\varepsilon$: $- \left.\dfrac{(\omega-\omega_0+i\varepsilon)\, e^{-i\omega(t-t')}}{(\omega-\omega_0+i\varepsilon)(\omega+\omega_0+i\varepsilon)}\right|_{\omega=\omega_0-i\varepsilon} =$

$$= -\frac{e^{-i(\omega_0-i\varepsilon)(t-t')}}{2\omega_0} \xrightarrow[\varepsilon\to 0]{} -\frac{e^{-i\omega_0(t-t')}}{2\omega_0}$$

Residue at $\omega = -\omega_0 - i\varepsilon$: $- \left.\dfrac{(\omega+\omega_0+i\varepsilon)\, e^{-i\omega(t-t')}}{(\omega-\omega_0+i\varepsilon)(\omega+\omega_0+i\varepsilon)}\right|_{\omega=-\omega_0-i\varepsilon} =$

$$= \frac{e^{i(\omega_0+i\varepsilon)(t-t')}}{2\omega_0} \xrightarrow[\varepsilon\to 0]{} \frac{1}{2\omega_0} e^{i\omega_0(t-t')}$$

$\therefore$ For $t > t'$, $G(t,t') = -i \dfrac{1}{2\omega_0}\left\{ e^{i\omega_0(t-t')} - e^{-i\omega_0(t-t')} \right\} =$

$$= \frac{1}{\omega_0}\sin[\omega_0(t-t')]$$

The Green's function is $G(t,t') = \begin{cases} 0 , & t < t' \\ \dfrac{1}{\omega_0}\sin[\omega_0(t-t')] , & t > t' \end{cases}$

Apply this to the driving force $f(t) = \begin{cases} 0 , & t < 0 \\ \gamma e^{-at}, & t > 0 \end{cases}$

$$x(t) = \frac{\gamma}{\omega_0}\int_0^t \sin[\omega_0(t-t')]\, e^{-at'} dt' = \frac{\gamma}{\omega_0}\,\mathrm{Im}\int_0^t e^{i\omega_0(t-t')} e^{-at'} dt' =$$

$$= \frac{\gamma}{\omega_0}\,\mathrm{Im}\left\{ e^{i\omega_0 t}\int_0^t e^{-(a+i\omega_0)t'} dt' \right\} = \frac{\gamma}{\omega_0}\,\mathrm{Im}\left\{ e^{i\omega_0 t}\left[-\frac{e^{-(a+i\omega_0)t'}}{a+i\omega_0}\right]_0^t \right\}$$

$$= \frac{\gamma}{\omega_0}\,\mathrm{Im}\left\{ \frac{e^{i\omega_0 t}\cdot(a-i\omega_0)}{a^2+\omega_0^2}\left(1 - e^{-(a+i\omega_0)t}\right) \right\} =$$

$$= \frac{\gamma}{\omega_0}\cdot\frac{1}{a^2+\omega_0^2}\,\mathrm{Im}\left\{ (a-i\omega_0)\Big(\underbrace{e^{i\omega_0 t}}_{\cos\omega_0 t + i\sin\omega_0 t} - e^{-at}\Big)\right\} =$$

$$= \frac{\gamma}{\omega_0}\cdot\frac{1}{a^2+\omega_0^2}\left\{ a\sin\omega_0 t - \omega_0\cos\omega_0 t + \omega_0 e^{-at}\right\}$$

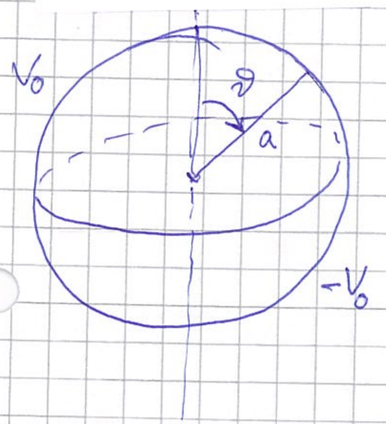
Check: $\sin\omega_0 t$ and $\cos\omega_0 t$ solve the homogeneous equation

Differentiating e^{-at} twice $\Rightarrow a^2 e^{-at}$ and $\dfrac{\gamma}{a^2+\omega_0^2}\left(a^2 e^{-at} + \omega_0^2 e^{-at}\right) = \gamma e^{-at}$

9. A conducting sphere of radius *a* is divided into two electrostatically separated hemispheres by a thin insulating barrier at its equator. The top hemisphere is maintained at a potential V_0, and the bottom hemisphere at $-V_0$. Show that the potential *exterior* to the two hemispheres is

$$V(r,\theta) = V_0 \sum_{s=0}^{\infty} (-1)^s (4s+3) \frac{(2s-1)!!}{(2s+2)!!} \left(\frac{a}{r}\right)^{2s+2} P_{2s+1}(\cos\theta)$$

where $P_{2s+1}(\cos\theta)$ is the Legendre polynomial of order *2s+1* (4p)



The potential is given by the Laplace equation $\nabla^2 \Psi = 0$. Use spherical polar coordinates centered at the origin of the sphere. After separating the variables we have (A, W&H 15,41):

$$\Psi(r,\vartheta,\varphi) = \sum_{\ell,m} \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1}\right) P_\ell^m(\cos\vartheta)\left(A'_{\ell m}\sin(m\varphi) + B'_{\ell m}\cos(m\varphi)\right)$$

There is no azimuthal dependence in the boundary condition so $m=0$. Furthermore, we are interested in the behavior exterior to the sphere, including $r\to\infty$, so $A_{\ell m}=0$. The A_{00} contribution only adds a constant to the potential which can be set to zero.

Thus $\Psi(r,\vartheta) = \sum_\ell B_\ell r^{-\ell-1} P_\ell(\cos\vartheta)$

$$\Psi(a,\vartheta) = \begin{cases} V_0, & 0 \le \vartheta < \frac{\pi}{2} \\ -V_0, & \frac{\pi}{2} < \vartheta \le \pi \end{cases}$$

Find the coefficients B_ℓ through projection:

$$\int_{-1}^{1} \Psi(a,x) P_n(x)\,dx = \sum_\ell B_\ell a^{-\ell-1} \int_{-1}^{1} P_\ell(x) P_n(x)\,dx =$$

$$= \sum_\ell B_\ell a^{-\ell-1} \frac{2}{2n+1}\delta_{n\ell} = B_n a^{-n-1} \cdot \frac{2}{2n+1}$$

$$\int_{-1}^{1} \Psi(a,x) P_n(x)\,dx = V_0 \left\{ \int_0^1 P_n(x)\,dx - \int_{-1}^{0} P_n(x)\,dx \right\} =$$

$$= V_0 \left\{ \int_0^1 P_n(x)\,dx + \int_1^0 P_n(-x)\,dx \right\} =$$

$$= V_0 \left\{ \int_0^1 P_n(x)dx - (-1)^n \int_0^1 P_n(x)dx \right\} = \begin{cases} 0, & n \text{ even} \\ 2V_0 \int_0^1 P_n(x)\,dx, & n \text{ odd} \end{cases}$$

Use the recurrence relation $P_n(x) = \frac{1}{2n+1}\left[P'_{n+1}(x) - P'_{n-1}(x)\right]$

Since n is odd $n \pm 1$ are even.

For all n we have $P_n(1) = 1$

For even n we have $P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$ (A, W&H 15, 11)

$$\therefore \int_0^1 P_{2k+1}(x)\,dx = \frac{1}{2(2k+1)+1}\left\{P_{2k}(0) - P_{2k+2}(0)\right\} =$$

$$= \frac{1}{4k+3}\left\{(-1)^k \frac{(2k-1)!!}{(2k)!!} - (-1)^{k+1}\frac{(2k+1)!!}{(2k+2)!!}\right\} =$$

$$= (-1)^k \cdot \frac{1}{4k+3} \cdot \frac{(2k-1)!!}{(2k)!!}\left\{1 + \frac{2k+1}{2k+2}\right\} =$$

$$= \frac{(-1)^k}{4k+3} \cdot \frac{(2k-1)!!}{(2k)!!}\left\{\frac{2k+2+2k+1}{2k+2}\right\} = (-1)^k \frac{(2k-1)!!}{(2k+2)!!}$$

Thus, $B_{2k+1} \cdot a^{-2k-2} \cdot \frac{2}{4k+3} = 2V_0(-1)^k \frac{(2k-1)!!}{(2k+2)!!}$

$$B_{2k+1} = V_0(-1)^k(4k+3)\frac{(2k-1)!!}{(2k+2)!!} \cdot a^{2k+2}$$

$$\therefore V(r, \vartheta) = V_0 \sum_{k=0}^{\infty} (-1)^k (4k+3) \frac{(2k-1)!!}{(2k+2)!!}\left(\frac{a}{r}\right)^{2k+2} P_{2k+1}(\cos\vartheta)$$

Alternative based on the generating function

Integrate with respect to x:

$$\int_0^1 \frac{dx}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n \int_0^1 P_n(x)dx$$

$$\int_0^1 \frac{dx}{\sqrt{1-2xt+t^2}} = \left[-\frac{\sqrt{1-2xt+t^2}}{t}\right]_0^1 = \frac{\sqrt{1+t^2} - (1-t)}{t} =$$

(9,3)

$$= -\frac{1}{t} + 1 + \sum_{n=0}^{\infty} \binom{1/2}{n} t^{2n-1} = \left\{\binom{1/2}{0} - 1\right\} \cdot \frac{1}{t} + 1 + \sum_{n=0}^{\infty} \binom{1/2}{n+1} t^{2n+1}$$

The coefficient $\binom{1/2}{0} = \frac{\Gamma(3/2)}{\Gamma(1)\Gamma(3/2)} = 1$ so the $\frac{1}{t}$ term vanish.

Uniqueness of power series implies that

$$\int_0^1 P_0(x)\,dx = 1, \quad \int_0^1 P_{2\ell}(x)\,dx = 0 \text{ and } \int_0^1 P_{2\ell+1}(x)\,dx = \binom{1/2}{\ell+1}$$

Pochhammer

and $$\binom{1/2}{\ell+1} = \frac{\left(\frac{1}{2} - (\ell+1) + 1\right)_{\ell+1}}{(\ell+1)!} = \frac{\left(\frac{1}{2} - \ell\right)_{\ell+1}}{(\ell+1)!} =$$

$$= \frac{\left(\frac{1}{2} - \ell\right)\left(\frac{1}{2} - \ell + 1\right) \cdots \left(\frac{1}{2}\right)}{(\ell+1)!} = \frac{(-1)^{\ell}\left(\frac{2\ell-1}{2}\right)\left(\frac{2\ell-3}{2}\right) \cdots \left(\frac{1}{2}\right)}{(\ell+1)!} =$$

$$= \frac{(-1)^{\ell}(2\ell-1)!!}{2^{\ell+1}(\ell+1)!} = (-1)^{\ell} \frac{(2\ell-1)!!}{(2\ell+2)!!}$$