FK7048 - Mathematical Methods in Physics
Exam 2018-11-02, 08:00-13:00

Allowed help:
the notes you made during the lectures/tutorials and the lecture/tutorial notes that are posted on the course website. The course book, Arfken, Weber, Harris: Mathematical Methods for Physicists. In case you use an electronic copy of the book and/or lecture/tutorial notes, the only program that you are allowed to use on your device is a pdf reader with the above mentioned material available. All other programs/apps should be closed. Mobile phones are under no circumstances allowed!
First read the whole exam, and start with the exercises you think you'll be able to best!
If you use a theorem in the solution of a problem, this should be stated explicitly. If you use a result from the book, you should give a clear reference, such as an equation number.
Good luck! Eddy Ardonne

1. The 2π-periodic function $f(x)$ is defined by specifying $f(x)$ on the interval $-\pi < x < \pi$ as $f(x) = 0$ for $-\pi < x \leq 0$ and $f(x) = x$ for $0 \leq x < \pi$.
   (a) (4p) Express $f(x)$ in terms of a Fourier series.
   (b) (1p) Show that $\frac{\pi^2}{8} = \sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2}$.
2. (5p) Use the contour integration method to evaluate the integral $\int_0^{+\infty} \frac{x \sin(x)}{x^2+a^2} dx$ where a is an arbitrary real parameter.
3. We consider the following differential equation:
   $$xy''(x) + (3-x)y'(x) - y(x) = 0 \ .$$
   (a) (1p) Determine all the singular points of this equation, and their nature.
   (b) (4p) Use Frobenius' method to determine the general solution of this differential equation.
4. We assume that the amplitude $u(r,t)$ of a circular drum with radius a does not depend on the polar angle ϕ. The edge of the drum is held fixed, $u(a,t) = 0$. The amplitude is described by the wave equation in polar coordinates,
   $$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} \ .$$
   (a) (4p) Find the general solution of the wave equation for $u(r,t)$, by separation of variables, for the given boundary condition. Denote the m^{th} zero of the Bessel function $J_\nu(x)$ by $\alpha_{m,\nu}$.
   (b) (1p) We now also impose the boundary conditions $u(r,0) = J_0(\alpha_{2,0} r/a)$ and $\left.\frac{\partial u(r,t)}{\partial t}\right|_{t=0} = 0$. Find the solution for this case.
5. Two identical, frictionless pendulums are coupled by a spring. It is given that the equations of motion, in the small amplitude limit, are given by ($\alpha = g/l$ and $\beta = k/m$)
   $$\ddot{x}(t) + \alpha x(t) + \beta(x(t) - y(t)) = 0 \qquad \ddot{y}(t) + \alpha y(t) - \beta(x(t) - y(t)) = 0 \ .$$
   The initial conditions are $x(0) = 0$, $\dot{x}(0) = v_0$, and $y(0) = 0$, $\dot{y}(0) = 0$.

(a) (4p) Solve the equations of motion for the given boundary conditions, using the Laplace transform method.

(b) (1p) Does, for *generic parameters*, the system come back to its configuration at $t = 0$? If so, at what time does this happen for the first time? If not, why not? *Hint*: start by looking at $\dot{x}(t)$.

6. The one-dimensional time-independent Schrödinger equation for the wave function $\psi(x)$ of a particle in a (finite) potential can be written as $-\psi''(x) + U(x)\ \psi(x) = \epsilon\ \psi(x)$ where $U(x)$ and ϵ are the rescaled potential and energy (i.e., in units of $\frac{\hbar^2}{2m}$).

   A bound state is a state of energy ϵ such that $\epsilon < \lim_{x\to-\infty} U(x)$ and $\epsilon < \lim_{x\to+\infty} U(x)$. It means that the particle is trapped in the potential and can not escape.

   (a) (0.5p) Justify (in words) that for bound states $\lim_{x\to-\infty} \psi(x) = 0$ and $\lim_{x\to+\infty} \psi(x) = 0$.

   (b) (1p) Show that the Hamiltonian $\mathcal{H} = -\frac{d^2}{dx^2} + U(x)$ is an Hermitian operator. That is, show that $\langle \mathcal{H}f, g\rangle = \langle f, \mathcal{H}g\rangle$ for any square integrable functions f and g vanishing at infinity.

   (c) (2.5) Let ψ_1 and ψ_2 be two bound-state solutions of the Schrödinger equation with energies ϵ_1 and ϵ_2 respectively. For real, but otherwise arbitrary a, b, show that:

   $$\Big[W(\psi_1, \psi_2)\Big]_a^b = (\epsilon_1 - \epsilon_2) \int_a^b \psi_1(x)\psi_2(x)dx \qquad (1)$$

   where $W(\psi_1, \psi_2)$ is the Wronskian of ψ_1 and ψ_2.

   (d) (1p) Show that for one-dimensional potentials, bound states can not be degenerate. This means that if two wave functions describe states with the same energy then they are proportional.

7. The set of polynomials $f_n(x)$ of degree $n \geq 0$ satisfy the following differential equation

   $$(x^2 - 4)y''(x) + 3xy'(x) - n(n+2)y(x) = 0\ .$$

   It is given that these polynomials have the following generating function $g(x,t) = \frac{1}{1-xt+t^2}$.

   (a) (1p) Determine $f_0(x)$, $f_1(x)$ and derive the recursion relation that expresses $f_n(x)$ in terms of $f_{n-1}(x)$ and $f_{n-2}(x)$.

   (b) (1p) Determine the parity of $f_n(x)$ and calculate $f_n(2)$.

   (c) (1p) Find the factor $w(x)$, such that the differential equation becomes self-adjoint. Exploit the freedom in $w(x)$ so that $w(0)$ is real (answer: $w(x) = \sqrt{4 - x^2}$).

   (d) (1p) On which interval are the polynomials $f_n(x)$ orthogonal for the weight factor $w(x)$? That is, determine a so that $\int_{-a}^{a} w(x)f_m(x)f_n(x)dx = 0$ if $m \neq n$.

   (e) (1p) Determine the normalization of the polynomials $f_n(x)$ for the weight factor $w(x)$. That is, determine $\int_{-a}^{a} w(x)\big(f_n(x)\big)^2 dx$.

In the exam, you are allowed to use that the following integral, with $-1 < t < 1$,

$$\frac{2}{\pi}\int_0^{\pi} \frac{\sin^2(\alpha)d\alpha}{(1 - 2\cos(\alpha)t + t^2)^p}$$

takes the values $1, (1-t^2)^{-1}, (1-t^2)^{-3}, (1+t^2)(1-t^2)^{-5}$ for $p = 1, 2, 3, 4$ respectively.

1) $f(x) = \begin{cases} 0 & -\pi < x \leq 0 \\ x & 0 \leq x < \pi \end{cases}$ and $f(x+2\pi) = f(x)$

a) Fourier series is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$,

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$ and

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

AWH (19.1 - 19.3).

So: $a_n = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx \overset{\text{P.I.}}{=} \underbrace{\frac{x}{\pi n}\sin(nx)\Big]_0^{\pi}}_{\text{gives } 0} - \frac{1}{\pi n}\int_0^{\pi}\sin(nx)\,dx$
$n > 0$

$$= \frac{1}{\pi n^2}\cos(nx)\Big]_0^{\pi} = \frac{1}{\pi n^2}\left[\cos(\pi n) - \cos(0)\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & n \text{ even} \\ \frac{-2}{\pi n^2} & n \text{ odd.} \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi}x^2\Big]_0^{\pi} = \frac{\pi}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx = -\frac{1}{\pi n}(x\cos nx)\Big|_0^{\pi} + \frac{1}{\pi n}\int_0^{\pi}\cos(nx)\,dx$$

$$= \frac{-1}{n}\cos(\pi n) + \underbrace{\frac{1}{\pi n^2}\sin(nx)\Big]_0^{\pi}}_{=0} = \frac{(-1)^{n+1}}{n}$$

So: $f(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{-2}{\pi(2k-1)^2}\cos((2k-1)x) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$

b) Evaluate $f(x)$ at $x=0$; this gives

$$f(0) = 0 = \frac{\pi}{4} + \sum_{k=1}^{\infty}\frac{-2}{\pi(2k-1)^2} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Alternative: use contour integration (much longer!).

2) Calculate $I = \int_0^\infty \frac{x \sin x \, dx}{x^2+a^2}$, for $a \in \mathbb{R}$.

Lets assume that $a > 0$.

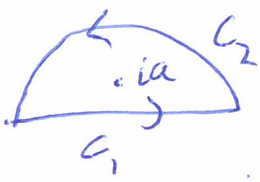
Integrant is even, so we write $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{x^2+a^2}$,

or $I = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+a^2} dx$.

Now, $\frac{z e^{iz}}{z^2+a^2}$ is analytic in UHP, (apart from one pole at ia)

$\lim_{|z|\to\infty} z/_{z^2+a^2} = 0$ in UHP, so we can use Jordan's Lemma:

$\int_{C_2} \frac{z e^{iz}}{z^2+a^2} dz = 0$.

Contour C: 

Thus: $I = \frac{1}{2} \operatorname{Im} \oint_C \frac{z e^{iz}}{z^2+a^2} dz = \frac{1}{2} \operatorname{Im}\left[2\pi i \operatorname{Res}_{z=+ia} \frac{z e^{iz}}{z^2+a^2}\right]$

Residue of $\frac{z e^{iz}}{z^2+a^2}$ (at $z=+ia$) is: $\lim_{z\to ia} \frac{\cancel{(z-ia)} z e^{iz}}{\cancel{(z-ia)}(z+ia)} = \frac{ia e^{-a}}{2ia} = \frac{e^{-a}}{2}$

So: $I = \frac{1}{2} \operatorname{Im}\left(\frac{2\pi i}{2} e^{-a}\right) = \frac{\pi}{2} e^{-a}$, where $a > 0$.

The integral is the same for a and $-a$, so for $a \neq 0$, we have: $I = \frac{\pi}{2} e^{-|a|}$.

For $a=0$: $I = \int_0^\infty \frac{\sin x}{x} dx$, which is the Dirichlet integral, as seen in class, and has the value $\pi/2$.

So, for arbitrary $a \in \mathbb{R}$, we have $I = \frac{\pi}{2} e^{-|a|}$.

Or:
$a=0$ case: view a^2 terms as a converging factor, and take $a \to 0$ limit.

3) ODE is: $x\, y''(x) + (3-x)\, y'(x) - y(x) = 0$.

a) Write $y''(x) + \left(\frac{3}{x} - 1\right) y'(x) - \frac{y(x)}{x} = 0$, so we have a <u>regular</u> singular point at $x = 0$,

because $P(x) = \frac{3}{x} - 1$ and $Q(x) = \frac{1}{x}$ diverge at $x = 0$, but $x P(x)$ and $x^2 Q(x)$ are finite at $x = 0$.

For ∞, we substitute $x = 1/z$, and we should consider

$$\frac{2z - P(1/z)}{z^2} \quad \text{and} \quad \frac{Q(1/z)}{z^4} \quad \text{at } z = 0$$ (7.22 + following discussion).

$\frac{2z - (3z - 1)}{z^2}$ diverges at $z = 0$ ~~and~~ faster than $1/z$, so we have an <u>irregular</u> singular point at $x \to \infty$.

b) Frobenius method: substitute $y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}$; $a_0 \neq 0$

$$y'(x) = \sum_{k=0}^{\infty} a_k (k+\alpha) x^{k+\alpha-1}; \quad y''(x) = \sum_{k=0} a_k (k+\alpha)(k+\alpha-1) x^{k+\alpha-2}.$$

This gives:

$$\sum_{k=0}^{\infty} a_k (k+\alpha)(k+\alpha-1) x^{k+\alpha-1} + 3 \sum_{k=0}^{\infty} a_k (k+\alpha) x^{k+\alpha-1}$$
$$- \sum_{k=0}^{\infty} a_k (k+\alpha) x^{k+\alpha} - \sum_{k=0}^{\infty} a_k \cancel{(k+\alpha)} x^{k+\alpha} = 0.$$

Lowest power of x: occurs for $k = 0$, is $x^{\alpha-1}$.

Coefficient: $a_0 (\cancel{k} + \alpha)(\alpha - 1) + 3 a_0 \alpha = 0$

So, as $a_0 \neq 0$, we have $\alpha^2 - \alpha + 3\alpha = \alpha(\alpha + 2) = 0$

$\Rightarrow \alpha = 0$ ~~and~~ or $\alpha = -2$.

[Start with $\alpha = 0$]

Gen. recursion relation:

$$\sum_{k=0}^{\infty} a_k (k+\alpha)(k+\alpha+2) x^{k+\alpha-1} - \sum_{k=0}^{\infty} a_k (k+\alpha+1) x^{k+\alpha} = 0$$

(~~for~~ 1st, 2nd term) (3rd, 4th term).

Rewrite last term:

$$\sum_{k=0}^{\infty} a_k (k+\alpha)(k+\alpha+2) x^{k+\alpha-1} - \sum_{k=1}^{\infty} a_{k-1} (k+\alpha) x^{k+\alpha-1} = 0$$

So we have: $a_k (k+\alpha)(k+\alpha+2) = a_{k-1}(k+\alpha)$

Use $\alpha = 0$ first: $k+\alpha = k$ is never zero.

$\Rightarrow$ $a_k = \dfrac{a_{k-1}}{k+2}$; so $a_1 = \dfrac{a_0}{3}$; $a_2 = \dfrac{a_0}{4 \cdot 3 \cdot \cancel{2}}$,

$a_3 = \dfrac{a_0}{5 \cdot 4 \cdot 3}$, or $a_k = \dfrac{2 a_0}{(k+2)!}$

So (set $a_0 = \frac{1}{2}$), $y(x) = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{(k+2)!} = \frac{1}{x^2} \sum_{k=0}^{\infty} \frac{x^{k+2}}{(k+2)!}$

$$= \frac{1}{x^2}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!} - 1 - x\right) = \frac{1}{x^2}\left(e^x - 1 - x\right)$$

Now, use $\alpha = -2$. Then: $a_1(1-2)(1-2+2) = a_0(1-2)$

So $a_1 = a_0$.

$a_2 \cdot (0)(2) = a_1 \cdot (0)$, so a_2 is not determined, so we can set it to zero! Thus $a_3 = a_4 = \dots = 0$ as well.

Thus: $y(x) = \dfrac{1}{x^2} + \dfrac{1}{x}$. We can write gen. solution as

$$y(x) = c_1 \frac{e^x}{x^2} + c_2\left(\frac{1}{x^2} + \frac{1}{x}\right)$$

4. a) We write $u(r,t) = R(r)T(t)$, and substitute in $\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$, which gives

$$\frac{1}{v^2} R(r)T''(t) = T(t)\left[R''(r) + \frac{1}{r}R'(r)\right], \text{ or}$$

$$\frac{1}{v^2}\frac{T''(t)}{T(t)} = \frac{R''(r)}{R(r)} + \frac{1}{rR(r)}R'(r),$$ which has to be constant (LHS function of t, RHS function of r).

We expect osc. solutions, so set $\frac{1}{v^2}\frac{T''(t)}{T(t)} = -\lambda^2$ and

$$\frac{R''(r)}{R(r)} + \frac{1}{rR(r)}R'(r) = -\lambda^2.$$

Solve eq^n for $T(t)$: $T''(t) = -\lambda^2 v^2 T(t)$, so

$$T(t) = A\cos(\lambda v t) + B\sin(\lambda v t)$$

The eq^n for $R(r)$ becomes: $r^2R''(r) + rR'(r) + \lambda^2 r^2 R(r) = 0$,

(scale w/ r^2)

~~Convenient to rescale ...~~ The solutions are Bessel functions $J_0(\lambda r)$, $Y_0(\lambda r)$, but $Y_0(\lambda r)$ is not regular at $r=0$, so we need $R(r) = J_0(\lambda r)$.

We need to satisfy $u(a,t) = 0$, so we need $R(a) = J_0(\lambda a) = 0$.

$\alpha_{m,0}$ is the m^{th} zero of $J_0(x)$, so we have $\lambda a = \alpha_m$,

so that we have the solution $R(r) = \sum_{m=1}^{\infty} c_m J_0\left(\frac{\alpha_m r}{a}\right)$.

Combining the solutions for $T(t)$ and $R(r)$, we have:

$$u(r,t) = \sum_{m=1}^{\infty} J_0\left(\frac{\alpha_m r}{a}\right)\left[A_m \cos\left(\frac{\alpha_m V t}{a}\right) + B_m \sin\left(\frac{\alpha_m V t}{a}\right)\right]$$

b) We also have $u(r,0) = J_0(\alpha_{2,0} r/a)$

and $\left.\frac{\partial u(r,t)}{\partial t}\right|_{t=0} = 0$.

$$\frac{\partial u(r,t)}{\partial t} = \sum_{m=1}^{\infty} J_0\left(\frac{\alpha_m r}{a}\right)\left(\frac{\alpha_m V}{a}\right)\left[-A_m \sin\left(\frac{\alpha_m V t}{a}\right) + B_m \cos\left(\frac{\alpha_m V t}{a}\right)\right]$$

$$\left.\frac{\partial u(r,t)}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty} J_0\left(\frac{\alpha_m r}{a}\right)\left(\frac{\alpha_m V}{a}\right) B_m \cancel{\cos} = 0$$

$$\Rightarrow B_m = 0$$

Thus $u(r,0) = \sum_{m=1}^{\infty} J_0\left(\frac{\alpha_m r}{a}\right) A_m = J_0\left(\frac{\alpha_{2,0} r}{a}\right)$, so

$A_2 = 1$, $A_m = 0$ $m \neq 2$. (using orthogonality).

This gives

$$u(r,t) = J_0\left(\frac{\alpha_{2,0} r}{a}\right) \cos\left(\frac{\alpha_{2,0} V t}{a}\right)$$

5 a) The EOM are:

$$\ddot{x}(t) + \alpha x(t) + \beta (x(t) - y(t)) = 0$$
$$\ddot{y}(t) + \alpha y(t) - \beta (x(t) - y(t)) = 0,$$

$x(0) = y(0) = 0$

$\dot{x}(0) = V$; $\dot{y}(0) = 0$

Use Laplace transform to find the solution:

$$\mathcal{L}(x(t)) = X(s); \mathcal{L}(y(t)) = Y(s)$$

$$\mathcal{L}(\ddot{x}(t)) = s^2 X(s) - s x(0) - \dot{x}(0)$$
$$= s^2 X(s) - V$$
$$\mathcal{L}(\ddot{y}(t)) = s^2 Y(s) - s y(0) - \dot{y}(0) = s^2 Y(s)$$

So we get $s^2 X(s) + \alpha X(s) + \beta X(s) - \beta Y(s) - V = 0$
$$s^2 Y(s) + \alpha Y(s) + \beta Y(s) - \beta (X(s)) = 0$$

Add eq^{n}s: $s^2 (X(s) + Y(s)) + \alpha (X(s) + Y(s)) = V$

subtract $s^2 (X(s) - Y(s)) + \alpha (X(s) - Y(s)) + 2\beta (X(s) - Y(s)) = V$

So $X(s) + Y(s) = \dfrac{V}{s^2 + \alpha}$; $X(s) - Y(s) = \dfrac{V}{s^2 + \alpha + 2\beta}$

~~So the inverse transform gives simple cosine function: (20.134)~~

~~$X(s) + Y(s) = V \cos(\sqrt{\alpha} t)$; $X(s) - Y(s) = V \cos(\sqrt{\alpha + 2\beta} t)$~~

~~Solution:~~

~~$X(s) = \frac{V}{2} [\cos(\sqrt{\alpha} t) +$~~

The inverse gives sin function 20:35:

$$X(s)+Y(s) = \frac{V}{\sqrt{\alpha}} \sin(\sqrt{\alpha}\, t)$$

$$X(s)-Y(s) = \frac{V}{\sqrt{\alpha+2\beta}} \sin(\sqrt{\alpha+2\beta}\, t), \text{ or}$$

$$x(t)\ \cancel{X(s)} = \frac{V}{2}\left[\frac{\sin\sqrt{\alpha}\, t}{\sqrt{\alpha}} + \frac{\sin\sqrt{\alpha+2\beta}\, t}{\sqrt{\alpha+2\beta}}\right]$$

$$y(t)\ \cancel{Y(s)} = \frac{V}{2}\left[\frac{\sin\sqrt{\alpha}\, t}{\sqrt{\alpha}} - \frac{\sin\sqrt{\alpha+2\beta}\, t}{\sqrt{\alpha+2\beta}}\right]$$

b) $\dot{x}(t) = \frac{V}{2}\left[\cos\sqrt{\alpha}\, t + \cancel{\sin} \cos\sqrt{\alpha+2\beta}\, t\right]$

$\dot{x}(0) = V$, so if ~~we~~ system comes back to $t=0$ conf. at later times, we need $\cos(\sqrt{\alpha}\, t) = \cos(\sqrt{\alpha+2\beta}\, t) = 1$, or $\sqrt{\alpha}\, t = 2\pi k$, $\sqrt{\alpha+2\beta}\, t = 2\pi k'$*, which gives

$\frac{\sqrt{\alpha}}{\sqrt{\alpha+2\beta}} = k/k'$; This does <u>not</u> happen for generic parameters α and β.

* with $k, k' \in \mathbb{Z}$

6 a) The particle can not escape to $\pm$ infinity, so the amplitude for the particle being there should be zero. ~~so~~, which means that $\lim_{|x|\to\infty} \psi(x) = 0$.

b) ~~$\langle \mathcal{H} f, g \rangle = \int f^*(x) g$~~

$f(x), g(x)$ can be taken real, then

$$\langle \mathcal{H} f, g \rangle = \int_{-\infty}^{\infty} \left(-\frac{d^2}{dx^2} f + U(x) f(x)\right) g(x)\, dx$$

$$\overset{PI}{=} \int_{-\infty}^{\infty} f(x) U(x) g(x) + \underbrace{\left[-\left(\frac{d}{dx} f(x)\right) g(x)\right]_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} \left(\frac{d}{dx} f(x)\right) \frac{d}{dx}(g(x))\, dx$$

$$= \int_{-\infty}^{\infty} f(x) U(x) g(x) + \underbrace{\left[f(x) \frac{d^2}{dx^2} g(x)\right]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} f(x) \frac{d^2}{dx^2} g(x)\, dx$$

$$= \langle f, \mathcal{H} g \rangle.$$

c) We have $-\psi_1'' + U\psi_1 = \varepsilon_1 \psi_1$; $-\psi_2'' + U\psi_2 = \varepsilon_2 \psi_2$.

Wronskian: $W = \psi_1 \psi_2' - \psi_1' \psi_2$, so

$$W' = \psi_1' \psi_2' + \psi_1 \psi_2'' - \psi_1'' \psi_2' - \psi_1' \psi_2$$
$$= \psi_1 \psi_2'' - \psi_1'' \psi_2.$$

~~As~~ Use Schrodinger eq^n:

$$W' = \psi_1 (U - \varepsilon_2)\psi_2 - (U - \varepsilon_1)\psi_1 \psi_2 = (\varepsilon_1 - \varepsilon_2)\psi_1 \psi_2.$$

Integrate between a, b:

$$W(\psi_1, \psi_2)\Big]_a^b = (\varepsilon_1 - \varepsilon_2) \int_a^b \psi_1(x) \psi_2(x)\, dx.$$

d) The relation in c) holds for arbitrary a, b. So, if $\varepsilon_1 = \varepsilon_2$, this means that W is identically zero. If $W = 0$, then ψ_1 and ψ_2 are not

lin. independent!

7 a) Use the generating function:

$$g(x,t) = \sum_{n=0}^{\infty} f_n(x)\, t^n = \frac{1}{1 - xt + t^2}$$

$$\frac{1}{1-xt+t^2} = \frac{1}{1-(xt-t^2)} = 1 + (xt - t^2) + (xt - t^2)^2 + \text{h.o.t}$$

$$= 1 + xt + (x^2-1)t^2 + O(t^3),$$

$f_0(x) = 1$; $f_1(x) = x$, $f_2(x) = x^2 - 1$ (last one not asked for).

To get recursion, take derivative w.r.t. t:

$$\frac{\partial}{\partial t} g(x,t) = \frac{-1}{(1-xt+t^2)^2}(-x+2t) = \sum_{n=0}^{\infty} n f_n(x)\, t^{n-1}$$

So, we get $\dfrac{(x-2t)}{(1-xt+t^2)} = (1-xt+t^2)\displaystyle\sum_{n=0}^{\infty} n f_n(x)\, t^{n-1}$, or

$$(x-2t)\sum_{n=0}^{\infty} f_n(x) t^n = (1-xt+t^2)\sum_{n=0}^{\infty} n f_n(x)\, t^{n-1}.$$

Thus
$$\underbrace{\sum_{n=0}^{\infty} x f_n(x) t^n}_{1\text{st}} - \underbrace{2\sum_{n=0}^{\infty} f_n(x) t^{n+1}}_{2\text{nd}} - \underbrace{\sum_{n=0}^{\infty} n f_n(x)\, t^{n-1}}_{3\text{rd}} + \underbrace{x\sum_{n=0}^{\infty} n f_n(x)\, t^n}_{4\text{th}} - \underbrace{\sum_{n=0}^{\infty} n f_n(x) t^{n+1}}_{5\text{th}} = 0$$

Make powers of t the same:

$$\sum_{n=0}^{\infty} \underbrace{x(n+1) f_n(x)}_{1\text{st}\ \&\ 4\text{th}} t^n + \sum_{n=1}^{\infty}\Big[\underbrace{-2 f_{n-1}(x)}_{2\text{nd}} \underbrace{- (n-1) f_{n-1}(x)}_{5\text{th}}\Big] t^n - \sum_{n=-1}^{\infty} (n+1) f_{n+1}(x) t^n = 0$$

So, the generic coef of t^n is: $(n+1)\, x f_n(x) - (n+1) f_{n-1}(x) - (n+1) f_{n+1}(x) = 0$

$$f_{n+1}(x) = x f_n(x) - f_{n-1}(x), \text{ or}$$

$$\boxed{f_n(x) = x f_{n-1}(x) - f_{n-2}(x)}$$

b) Parity of f_0 is $+1$, of $f_1(x)$ = -1, $f_2(x)=+1$.

~~$f_2(x)$~~

One can show that $f_n(x)$ is even for n even, and odd for n odd using induction, w/ recursion.

Or: $g(-x,-t) = \sum_{n=0}^{\infty} f_n(-x)(-1)^n t^n$

$\parallel$

$g(x,t) = \sum_n f_n(x)\, t^n$, so $f_n(-x) = (-1)^n f_n(x)$.

$f_0(2)=1$; $f_1(2) = 2$; $f_2(2) = 4^2-1 = 3$.

So, assume $f_p(2) = p+1$ ~~up to~~ for $p=1,2,\dots,n$

Then $f_{n+1}(2) = 2f_n(2) - f_{n-1}(2)$

$= 2(n+1) - n = n+2$.

By induction, it follows that $f_n(2) = n+1$ for all n.

c) $W(x)$ is prop. to $\frac{1}{p_0} \exp\left[\int^x \frac{p_1(y)}{p_0(y)} dy\right]$

$$\int^x \frac{p_1(y)}{p_0(y)} dy = \int^x \frac{3y\,dy}{y^2-4} = \frac{3}{2}\int \frac{2y}{y^2-4} dy = \frac{3}{2}\log(y^2-4)$$

$= \log(y^2-4)^{3/2}$, so

$$\frac{1}{p_0} e^{\int^x \frac{p_1(y)}{p_0(y)} dy} = \frac{(x^2-4)^{3/2}}{x^2-4} = \sqrt{x^2-4}.$$

We want $w(x)$ to be real for $x=0$, so we set $w(x) = \sqrt{4-x^2}$.

d) We have a Sturm-Liouville problem, with weight factor $w(x) = \sqrt{4-x^2}$.

The orthogonality of S.L. solutions says that

$$\cancel{\lambda_u}\ (\lambda_u - \lambda_v)\int_a^b v^*(x)\,u(x)\,w(x)\,dx = \Big[w(x)\,p_0(x)\,(v^* u' - (v^*)' u)\Big]_a^b$$

AWH (8.20),

where $u(x)$ and $v(x)$ are solutions w/ eigenvalues λ_u and λ_v. Here $\lambda_n = n(n+2)$, so for $n \neq m$, we have that

$$\int_a^b f_m(x) f_n(x)\,w(x)\,dx = \frac{1}{(\lambda_m - \lambda_n)}\Big[w(x)\,p(x) \cdot (f_m' f_n - f_m f_n')\Big]_a^b$$

Because $w(x)$ is zero for $x = \pm 2$, we have (or $p_0(x)$)

$$\int_{-2}^{2} w(x) f_m(x) f_n(x)\,dx = 0 \text{ if } m \neq n.$$

e) The trick is to square the generating function and integrate it against $w(x)$.

$$\int_{-2}^{2} \frac{\sqrt{4-x^2}\,dx}{(1-xt+t^2)^2} = \sum_{m,n}\int_{-2}^{2} w(x)\,f_n(x)\,f_m(x)\,t^{n+m}\,dx$$

$$\underset{\text{orthogonality}}{=} \sum_m \int_{-2}^{2} w(x)\,[f_m(x)]^2\,t^{2m}\,dx$$

In the LHS, set $x = 2\cos\alpha$, so $dx = -2\sin\alpha\,d\alpha$.

~~$x = 2$~~ $\alpha = 0 \rightarrow x = 2$; $\alpha = \pi \rightarrow x = -2$ $\quad\sqrt{4-x^2} = 2|\sin\alpha| = 2\sin\alpha \quad (0 \le \alpha \le \pi)$

$$-\int_{\pi}^{0} \frac{4\sin^2\alpha\,d\alpha}{(1-2\cos\alpha\,t + t^2)^2} = \cancel{4\pi}\ 2\pi/(1-t^2) = 2\pi\sum_{n=0}^{\infty} t^{2n}$$

It follows that

$$\int_{-2}^{2} w(x)\, f_n^2(x)\, dx = 2\pi \quad \text{for } n \geq 0.$$

Note: the exam turned out to be too long. (scores were rescaled).