12.9 Material data

The following material properties may be used *only* when solving exercise problems. For a real design, data should be taken from latest official standard and *not* from this table (two values for the same material means different qualities).[1]

Material	Young's modulus E GPa	ν –	$\alpha 10^6$ K^{-1}	Ultimate strength MPa	Yield limit tension/ compression MPa	bending MPa	torsion MPa
Carbon steel							
141312-00	206	0.3	12	360 460	>240	260	140
141450-1	205	0.3		430 510	>250	290	160
141510-00	205	0.3		510 640	>320		
141550-01	205	0.3		490 590	>270	360	190
141650-01	206	0.3	11	590 690	>310	390	220
141650	206	0.3		860	>550	610	
					Offset yield strength $R_{p0.2}$ ($\sigma_{0.2}$)		
Stainless steel							
2337-02	196	0.29	16.8	>490	>200		
Aluminium							
SS 4120-02	70		23	170 215	>65		
SS 4120-24	70		23	220 270	>170		
SS 4425-06	70		23	>340	>270		

[1] Data in this table has been collected from B Sundström (editor): Handbok och Formelsamling i Hållfasthetslära, Institutionen för hållfasthetslära, KTH, Stockholm, 1998.

M

Mathematical Formulae

1 Mathematical Constants

$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69\ \ldots$

$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47\ \ldots$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$\gamma = 0.57721\ 56649\ \ldots$ = Euler's constant =

$$= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n\right)$$

$\phi = 1.61803\ 39887\ 49894\ 8482\ldots$ = Golden ratio = $(1+\sqrt{5})/2$

Values for some commonly used expressions

$\sqrt{2} = 1.4142$	$\lg 2 = 0.3010$	$\ln 2 = 0.6931$
$\sqrt{3} = 1.7320$	$\lg 3 = 0.4771$	$\ln 3 = 1.0986$
$\sqrt{5} = 2.2361$	$\lg 5 = 0.6990$	$\ln 5 = 1.6094$
$\sqrt{10} = 3.1623$	$\lg 10 = 1$	$\ln 10 = 2.3026$
$\sqrt{\pi} = 1.7725$	$\lg \pi = 0.4971$	$\ln \pi = 1.1447$
$\sqrt{e} = 1.6487$	$\lg e = 0.4343$	$\ln e = 1$

$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2} = 0.5$ $\qquad \sqrt[3]{2} = 1.2599$

$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.7071$ $\qquad \sqrt[3]{3} = 1.4422$

$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = 0.8660$ $\qquad \sqrt[3]{10} = 2.1544$

Feigenbaum numbers for the onset of chaos

$\alpha = 2.50290\ 7875\ \ldots$

$\delta = 4.66920\ 1609\ \ldots$

More information:
http://dgleahy.com/dgl/p15.html

Some exact values of trigonometric functions

degrees	radians	sin	cos	tan	cot
0	0	0	1	0	$\mp\infty$
15	$\frac{\pi}{12} \approx 0.2618$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$
30	$\frac{\pi}{6} \approx 0.5236$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45	$\frac{\pi}{4} \approx 0.7854$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
60	$\frac{\pi}{3} \approx 1.0472$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
75	$\frac{5\pi}{12} \approx 1.3090$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$
90	$\frac{\pi}{2} \approx 1.5708$	1	0	$\pm\infty$	0
105	$\frac{7\pi}{12} \approx 1.8326$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$\sqrt{3}-2$
120	$\frac{2\pi}{3} \approx 2.0944$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
135	$\frac{3\pi}{4} \approx 2.3562$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	-1
150	$\frac{5\pi}{6} \approx 2.6180$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$
165	$\frac{11\pi}{12} \approx 2.8798$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\sqrt{3}-2$	$-(2+\sqrt{3})$
180	$\pi \approx 3.1416$	0	-1	0	$\mp\infty$
195	$\frac{13\pi}{12} \approx 3.4034$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$
210	$\frac{7\pi}{6} \approx 3.6652$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
225	$\frac{5\pi}{4} \approx 3.9270$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	1
240	$\frac{4\pi}{3} \approx 4.1888$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
255	$\frac{17\pi}{12} \approx 4.4506$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$
270	$\frac{3\pi}{2} \approx 4.7124$	-1	0	$\pm\infty$	0
285	$\frac{19\pi}{12} \approx 4.9742$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$\sqrt{3}-2$
300	$\frac{5\pi}{3} \approx 5.2360$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
315	$\frac{7\pi}{4} \approx 5.4978$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	-1
330	$\frac{11\pi}{6} \approx 5.7596$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$
345	$\frac{23\pi}{12} \approx 6.0214$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\sqrt{3}-2$	$-(2+\sqrt{3})$
360	$2\pi \approx 6.2832$	0	1	0	$\mp\infty$

2 Algebra

Some identities

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ (binomial coefficients)

$a^2 - b^2 = (a-b)(a+b)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Second degree equation

$ax^2 + bx + c = 0$ $\qquad$ $x^2 + px + q = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\qquad$ $x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$

$x_1 + x_2 = -p \qquad x_1 \cdot x_2 = q$

Arithmetic series

$t_i = t_1 + (i-1)\,d$ $\qquad$ $s_n = \sum_{i=1}^{n} t_i = n\,\frac{t_1 + t_n}{2}$

Geometric series

$t_i = t_1\,k^{i-1}$ $\qquad$ $s_n = t_1 \sum_{i=0}^{n-1} k^i = \frac{t_1(1-k^n)}{1-k}$

Logarithms

$\log_a x = \frac{\log_c x}{\log_c a} = \frac{\ln x}{\ln a}$

$\log xy = \log x + \log y$

$\log (x/y) = \log x - \log y$

$\log x^n = n \log x$

Factorial and semifactorials

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ $\qquad$ $0! = 1$

$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$

$(2n)!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n$

Stirling's formula for $n \gg 1$

$n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1 \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$\ln n! \approx n \ln n - n + \frac{1}{2} \ln 2\pi n$

Compound interest

Compound amount at the end of n years when a principal P is deposited

$A = P(1+r)^n$ $\qquad$ r = interest rate (in decimals) compounded annually

Accumulated amount at the end of n years when a principal P is deposited at the end of each year (amount of an annuity)

$A = P\frac{(1+r)^n - 1}{r}$

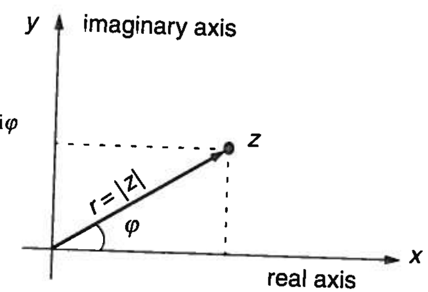
Present amount of an annuity in which the yearly payment at the end of each of n years is P

$A = P\frac{1-(1+r)^{-n}}{r}$

Complex numbers

$z = x + iy = r(\cos\varphi + i\sin\varphi) = r\,e^{i\varphi}$

$\arg z = \varphi + n\,2\pi = \arctan\frac{y}{x}$



$|z| = \sqrt{x^2 + y^2}$

$|z\,w| = |z| \cdot |w|$ $\qquad$ $\arg(z\,w) = \arg z + \arg w$ $\qquad$ See also Sec. M–4

$|z/w| = |z|/|w|$ $\qquad$ $\arg(z/w) = \arg z - \arg w$

Complex conjugate

$z^* = x - iy$

de Moivre formula

$z^n = r^n(\cos n\varphi + i\sin n\varphi)$

Cauchy-Schwarz inequality

$|a_1b_1 + a_2b_2 + \ldots + a_nb_n|^2 \le (|a_1|^2 + |a_2|^2 + \ldots + |a_n|^2)(|b_1|^2 + |b_2|^2 + \ldots + |b_n|^2)$

The equality holds if and only if $a_1/b_1 = a_2/b_2 = \ldots = a_n/b_n$

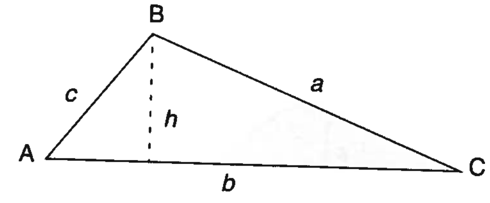
Prime number factorization of odd numbers

	0	100	200	300	400	500	600	700	800	900
1	–	–	3·67	7·43	–	3·167	–	–	$3^2\cdot89$	17·53
3	–	–	7·29	3·101	13·31	–	32·67	19·37	11·73	3·7·43
5	–	3·5·7	5·41	5·61	$3^4\cdot5$	5·101	$5\cdot11^2$	3·5·47	5·7·23	5·181
7	–	–	$3^2\cdot23$	–	11·37	$3\cdot13^2$	–	7·101	3·269	–
9	3^2	–	11·19	3·103	–	–	3·7·29	–	–	$3^2\cdot101$
11	–	3·37	–	–	3·137	7·73	13·47	$3^2\cdot79$	–	–
13	–	–	3·71	–	7·59	$3^3\cdot19$	–	23·31	3·271	11·83
15	3·5	5·23	5·43	$3^2\cdot5\cdot7$	5·83	5·103	3·5·41	5·11·13	5·163	3·5·61
17	–	$3^2\cdot13$	7·31	–	3·139	11·47	–	3·239	19·43	7·131
19	–	7·17	3·73	11·29	–	3·173	–	–	$3^2\cdot7\cdot13$	–
21	3·7	11^2	13·17	3·107	–	–	$3^3\cdot23$	7·103	–	3·307
23	–	3·41	–	17·19	$3^2\cdot47$	–	7·89	3·241	–	13·71
25	5^2	5^3	$3^2\cdot5^2$	$5^2\cdot13$	$5^2\cdot17$	$3\cdot5^2\cdot7$	5^4	$5^2\cdot29$	$3\cdot5^2\cdot11$	$5^2\cdot37$
27	3^3	–	–	3·109	7·61	17·31	3·11·19	–	–	$3^2\cdot103$
29	–	3·43	–	7·47	3·11·13	23^2	17·37	3^6	–	–
31	–	–	3·7·11	–	–	$3^2\cdot59$	–	17·43	3·277	$7^2\cdot19$
33	3·11	7·19	–	$3^2\cdot37$	–	13·41	3·211	–	$7^2\cdot17$	3·311
35	5·7	$3^3\cdot5$	5·47	5·67	3·5·29	5·107	5·127	$3\cdot5\cdot7^2$	5·167	5·11·17
37	–	–	3·79	–	19·23	3·179	$7^2\cdot13$	11·67	$3^3\cdot31$	–
39	3·13	–	–	3·113	–	$7^2\cdot11$	$3^2\cdot71$	–	–	3·313
41	–	3·47	–	11·31	$3^2\cdot7^2$	–	–	3·13·19	29^2	–
43	–	11·13	3^5	7^3	–	3·181	–	–	3·281	23·41
45	$3^2\cdot5$	5·29	$5\cdot7^2$	3·5·23	5·89	5·109	3·5·43	5·149	$5\cdot13^2$	$3^3\cdot5\cdot7$
47	–	$3\cdot7^2$	13·19	–	3·149	–	–	$3^2\cdot83$	$7\cdot11^2$	–
49	7^2	–	3·83	–	–	$3^2\cdot61$	11·59	7·107	3·283	13·73
51	3·17	–	–	$3^3\cdot13$	11·41	19·29	3·7·31	–	23·37	3·317
53	–	$3^2\cdot17$	11·23	–	3·151	7·79	–	3·251	–	–
55	5·11	5·31	3·5·17	5·71	5·7·13	3·5·37	5·131	5·151	$5\cdot3^2\cdot19$	5·191
57	3·19	–	–	3·7·17	–	–	$3^2\cdot73$	–	–	3·11·29
59	–	3·53	7·37	–	$3^3\cdot17$	13·43	–	3·11·23	–	7·137
61	–	7·23	$3^2\cdot29$	19^2	–	3·11·17	–	–	3·7·41	31^2
63	$3^2\cdot7$	–	–	3·121	–	–	3·13·17	7·109	–	$3^2\cdot107$
65	5·13	3·5·11	5·53	5·73	3·5·31	5·113	5·7·19	$3^2\cdot5\cdot17$	5·173	5·193
67	–	–	3·89	–	–	$3^4\cdot7$	23·29	13·59	$3\cdot17^2$	–
69	3·23	13^2	–	$3^2\cdot41$	7·67	–	3·223	–	11·79	3·17·19
71	–	$3^2\cdot19$	–	7·53	3·157	–	11·61	3·257	13·67	–
73	–	–	3·7·13	–	11·43	3·191	–	–	$3^2\cdot97$	7·139
75	$3\cdot5^2$	$5^2\cdot7$	$5^2\cdot11$	$3\cdot5^3$	$5^2\cdot19$	$5^2\cdot23$	$3^3\cdot5^2$	$5^2\cdot31$	53·7	$3\cdot5^2\cdot13$
77	7·11	3·59	–	13·29	$3^2\cdot53$	–	–	3·7·37	–	–
79	–	–	$3^2\cdot31$	–	–	3·193	7·97	19·41	3·293	11·89
81	3^4	–	–	3·127	13·37	7·83	3·227	11·71	–	$3^2\cdot109$
83	–	3·61	–	–	3·7·23	11·53	–	$3^3\cdot29$	–	–
85	5·17	5·37	3·5·19	5·7·11	5·97	$3^2\cdot5\cdot13$	5·137	5·157	3·5·59	5·197
87	3·29	11·71	7·41	$3^2\cdot43$	–	–	3·229	–	–	3·7·47
89	–	$3^3\cdot7$	17^2	–	3·163	19·31	13·53	3·263	7·127	23·43
91	7·13	–	3·97	17·23	–	3·197	–	7·113	$3^4\cdot11$	–
93	3·31	–	–	3·131	17·29	–	$3^2\cdot7\cdot11$	13·61	19·47	3·331
95	5·19	3·5·13	5·59	5·79	$3^2\cdot5\cdot11$	5·7·17	5·139	3·5·53	5·179	5·199
97	–	–	$3^3\cdot11$	–	7·71	3·199	17·41	–	3·13·23	–
99	$3^2\cdot11$	–	13·23	3·7·19	–	–	3·233	17·47	29·31	$3^3\cdot37$
	0	**100**	**200**	**300**	**400**	**500**	**600**	**700**	**800**	**900**

3 Geometric Formulae

A = area
P = perimeter
V = volume
S = surface area

Triangle



$$A + B + C = 180°$$

$$A = \frac{1}{2}\, bh = \frac{1}{2}\, ab \sin C = \sqrt{s(s-a)(s-b)(s-c)}$$

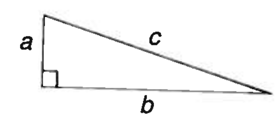
$$\text{where } s = \frac{1}{2}\,(a+b+c) = \frac{1}{2}\, P$$

$$a^2 = b^2 + c^2 - 2\,bc \cos A \qquad \text{(law of cosines)}$$

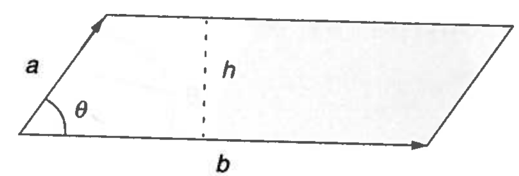
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{(law of sines)}$$

Right triangle, Theorem of Pythagoras

$$c^2 = a^2 + b^2$$

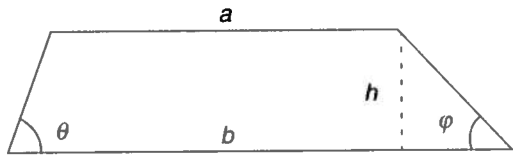


Parallelogram



$$A = bh = ab \sin \theta = |\mathbf{a} \times \mathbf{b}|$$

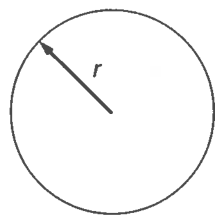
Trapezoid



$$A = \frac{1}{2}\, h\,(a + b)$$

$$P = a + b + h\left(\frac{1}{\sin\theta} + \frac{1}{\sin\varphi}\right)$$

Circle



$$A = \pi r^2$$

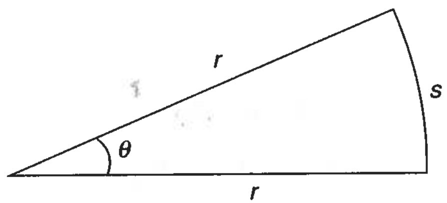
$$P = 2\pi r$$

Equation in rectangular coordinates

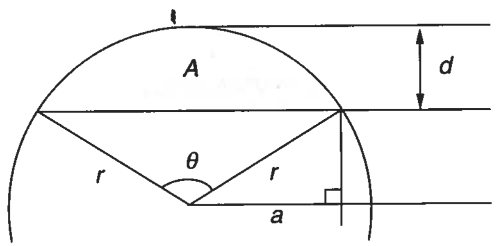
$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Sector of circle

$$A = \frac{1}{2}\, sr = \frac{1}{2}\, r^2\,\theta$$



Segment of circle

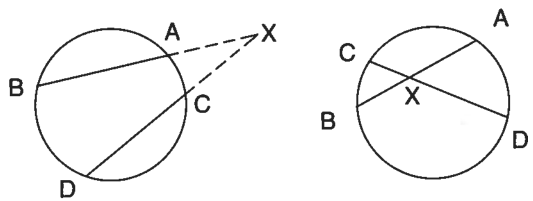


$$A = \frac{1}{2}\, r^2\,(\theta - \sin\theta)$$

$$a^2 = d(2r - d)$$

Theorem of chords

XA · XB = XC · XD

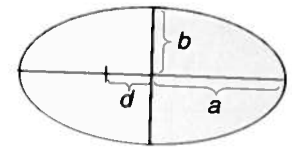


Ellipse

$$A = \pi ab$$

$$P \approx 2\pi\sqrt{\frac{1}{2}\,(a^2 + b^2)}$$

$$e = d/a$$

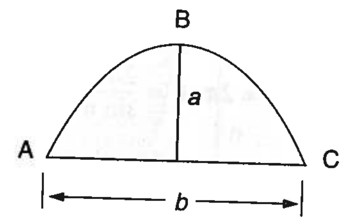


e = excentricity
d = distance from centre to foci

Equation in rectangular coordinates

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

Segment of a parabola



$$A = \frac{2}{3}\, ab$$

Length of arc ABC =

$$= \frac{1}{2}\sqrt{b^2 + 16a^2} + \frac{b^2}{8a}\,\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$

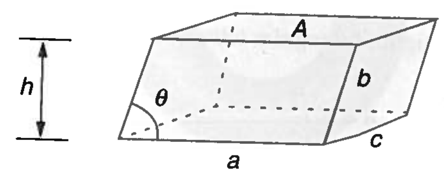
Equation in rectangular coordinates

$$(y - y_0) = \frac{1}{4d}\,(x - x_0)^2$$

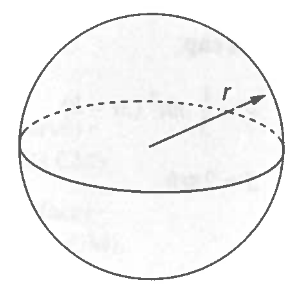
vertex at (x_0, y_0) and
focus at $(x_0, y_0 + d)$

Parallelepiped

$$V = Ah = abc\,\sin\theta$$



Sphere



$$V = \frac{4}{3}\,\pi r^3$$

$$S = 4\pi r^2$$

Equation in rectangular coordinates

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Shortest distance on the Earth's surface between two points P_1 *and* P_2 *(Great circle distance)*

$$d = \frac{2\pi\Omega}{360}R \qquad \cos\Omega = \sin v_1 \sin v_2 + \cos v_1 \cos v_2 \cos(u_1 - u_2)$$

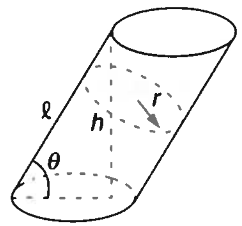
Ω = angle (in degrees) between points P_1 and P_2 measured at the Earth's centre
u_i = degree of longitude (E – W) of point P_i
v_i = degree of latitude (N – S) of point P_i
R = radius of the Earth

Circular cylinder

$$V = \pi r^2 h = \pi r^2 \ell \sin\theta$$

$$S = 2\pi r \ell = \frac{2\pi r h}{\sin\theta}$$

(lateral surface)

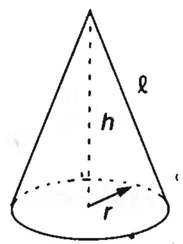


Circular cone

$$V = \frac{1}{3}\pi r^2 h$$

$$S = \pi r\sqrt{r^2 + h^2} = \pi r \ell$$

(lateral surface)

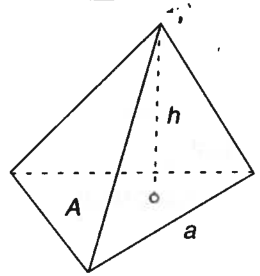


Pyramid

$$V = \frac{1}{3} A h$$

Height of a regular tetrahedron

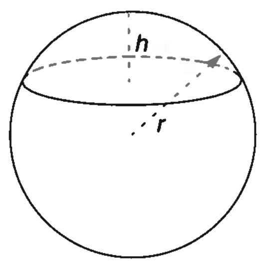
$$h = a\sqrt{\frac{2}{3}}$$



Spherical cap

$$V = \frac{1}{3}\pi h^2 (3r - h)$$

$$S = 2\pi r h$$

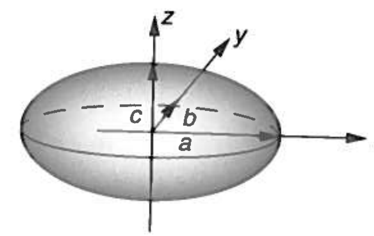


Ellipsoid

$$V = \frac{4}{3}\pi a b c$$

Equation in rectangular coordinates

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1$$

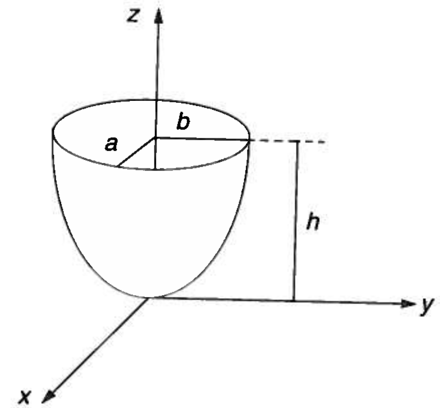


Elliptic paraboloid

$$V = \frac{1}{2}\pi a b h$$

Equation in rectangular coordinates

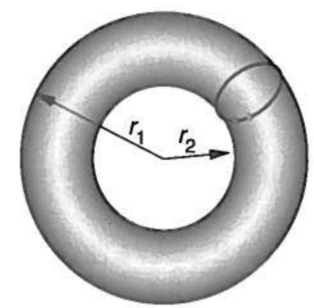
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{h}$$



Torus

$$V = \frac{1}{4}\pi^2 (r_1 + r_2)(r_1 - r_2)^2$$

$$S = \pi^2 (r_1{}^2 - r_2{}^2)$$



Pappus's centroid theorem (Guldin's rules)

Area of rotational surface = (length of generating curve) · (distance traversed by its CM)

Volume of rotational body = (area of generating surface) · (distance traversed by its CM).

4 Trigonometric Identities

Basic identities

$\tan \alpha = \dfrac{\sin \alpha}{\cos \alpha}$ $\quad \sec \alpha = \dfrac{1}{\cos \alpha}$

$\cot \alpha = \dfrac{\cos \alpha}{\sin \alpha} = \dfrac{1}{\tan \alpha}$ $\quad \csc \alpha = \dfrac{1}{\sin \alpha}$

$\sin^2 \alpha + \cos^2 \alpha = 1$

$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$

$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$

$\tan(\alpha \pm \beta) = \dfrac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$

$\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3\alpha$

$\cos 3\alpha = 4 \cos^3\alpha - 3 \cos \alpha$

$\sin^2 \dfrac{\alpha}{2} = \dfrac{1}{2}(1 - \cos \alpha)$ $\quad \tan \dfrac{\alpha}{2} = \dfrac{\sin \alpha}{1 + \cos \alpha} = \pm\sqrt{\dfrac{1 - \cos \alpha}{1 + \cos \alpha}}$

$\cos^2 \dfrac{\alpha}{2} = \dfrac{1}{2}(1 + \cos \alpha)$ $\quad \tan 2\alpha = \dfrac{2 \tan \alpha}{1 - \tan^2\alpha}$

$a \sin \alpha + b \cos \alpha = m \sin(\alpha + \varphi)$ $\quad m = \sqrt{a^2 + b^2}$

$\tan\varphi = \dfrac{b}{a}$

$\sin \alpha + \sin \beta = 2 \sin \dfrac{1}{2}(\alpha + \beta) \cos \dfrac{1}{2}(\alpha - \beta)$

$\sin \alpha - \sin \beta = 2 \cos \dfrac{1}{2}(\alpha + \beta) \sin \dfrac{1}{2}(\alpha - \beta)$

$\cos \alpha + \cos \beta = 2 \cos \dfrac{1}{2}(\alpha + \beta) \cos \dfrac{1}{2}(\alpha - \beta)$

$\cos \alpha - \cos \beta = -2 \sin \dfrac{1}{2}(\alpha + \beta) \sin \dfrac{1}{2}(\alpha - \beta)$

Euler's identities

$\sin \alpha = \dfrac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$ $\quad i = \sqrt{-1}$

$\cos \alpha = \dfrac{1}{2}(e^{i\alpha} + e^{-i\alpha})$ $\quad e^{i\alpha} = \cos \alpha + i \sin \alpha$

Complex numbers

$x + i\,y = r(\cos \alpha + i \sin \alpha)$ $\quad r = \sqrt{x^2 + y^2}$

$\tan \alpha = \dfrac{y}{x}$

See also Sec. M – 2

Hyperbolic functions

$\sinh x = \dfrac{1}{2}(e^x - e^{-x})$ $\quad \tanh x = \dfrac{\sinh x}{\cosh x}$

$\cosh x = \dfrac{1}{2}(e^x + e^{-x})$ $\quad \coth x = \dfrac{\cosh x}{\sinh x}$

5 Derivatives

General rules

$$f = u + v \Rightarrow f' = u' + v'$$
$$f = uv \Rightarrow f' = u'v + uv'$$
$$f = \frac{u}{v} \Rightarrow f' = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{where } y = F(u),\ u = f(x)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \ldots \quad \text{where } f = f(x, y, \ldots)$$

Logarithmic differentiation

$$f = u^a v^b w^c \ldots$$
$$\ln f = a \ln u + b \ln v + c \ln w + \ldots$$
$$\frac{df}{f} = a\frac{du}{u} + b\frac{dv}{v} + c\frac{dw}{w} + \ldots$$

Lagrange's method

Find maximum and minimum of the function $f(x_1 \ldots x_n)$ when

$$g_1(x_1 \ldots x_n) = 0$$
$$g_2(x_1 \ldots x_n) = 0$$
$$\vdots$$
$$g_m(x_1 \ldots x_n) = 0$$

Solutions are found by solving a system of $n + m$ equations

$$\frac{\partial F}{\partial x_i} = 0 \text{ for } i = 1, 2, \ldots, n \text{ and } g_j = 0 \text{ for } j = 1, 2, \ldots, m$$

where

$$F = f + \lambda_1 g_1 + \lambda_2 g_2 + \ldots + \lambda_m g_m$$

Special derivatives

function $f(x)$		derivative $\frac{df}{dx}$
x^n		$n x^{n-1}$
$\frac{1}{x}$		$-\frac{1}{x^2}$
$\sqrt{x}$		$\frac{1}{2\sqrt{x}}$
$\ln x$	$(x > 0)$	$\frac{1}{x}$
e^x		e^x
$\sin x$		$\cos x$
$\cos x$		$-\sin x$
$\tan x$		$1 + \tan^2 x = \frac{1}{\cos^2 x}$
$\cot x$		$-(1 + \cot^2 x) = -\frac{1}{\sin^2 x}$
a^x	$(a > 0)$	$a^x \ln a$
$\log_a x$	$(a, x > 0, a \neq 1)$	$\frac{1}{x}\log_a e = \frac{1}{x \ln a}$
$\arcsin x$		$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$		$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$		$\frac{1}{1+x^2}$

6 Integrals

General rules

$$\int_a^b (u+v)\,dx = \int_a^b u\,dx + \int_a^b v\,dx$$

$$\int_a^b u\,dx = \int_a^c u\,dx + \int_c^b u\,dx$$

$$\int_a^b u(x)\,dx = \int_{v(a)}^{v(b)} u(v)\,\frac{dx}{dv}\,dv = \int_{v(a)}^{v(b)} u(v)\,\frac{1}{\frac{dv}{dx}}\,dv \qquad v = v(x)$$

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt$$

$$\int_a^b u(x)\,v(x)\,dx = U(b)v(b) - U(a)v(a) - \int_a^b U(x)\,v'(x)\,dx$$

$U(x)$ = primitive function to $u(x)$

$$\int_a^b u\,v^{(n)}\,dx = [uv^{(n-1)} - u'v^{(n-2)} + u''v^{(n-3)} - \ldots]_a^b + (-1)^n \int_a^b v\,u^{(n)}\,dx$$

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(x,y)\,dy = f(x,h(x))h'(x) - f(x,g(x))g'(x) + \int_{g(x)}^{h(x)} \frac{\partial}{\partial x} f(x,y)\,dy$$

Cauchy-Schwarz inequality

$$\left|\int_a^b f(x)\,g(x)\,dx\right|^2 \le \left\{\int_a^b |f(x)|^2\,dx\right\}\left\{\int_a^b |g(x)|^2\,dx\right\}$$

Indefinite integrals (constants are omitted)

function	primitive function	
x^n	$\frac{1}{n+1}\,x^{n+1}$	$n \neq -1$

function	primitive function	
$\frac{1}{x-a}$	$\ln\lvert x-a\rvert$	$x \neq a$
$\frac{f'(x)}{f(x)}$	$\ln f(x)$	
a^x	$\frac{a^x}{\ln a}$	$(a > 0,\ a \neq 1)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\frac{\lvert x-a\rvert}{\lvert x+a\rvert}$	$\lvert x\rvert \neq a$
$\frac{1}{(ax+b)(px+q)}$	$\frac{1}{bp-aq}\ln\frac{\lvert px+q\rvert}{\lvert ax+b\rvert}$	
$\frac{x}{(ax+b)(px+q)}$	$\frac{1}{bp-aq}\left(\frac{b}{a}\ln\lvert ax+b\rvert - \frac{q}{p}\ln\lvert px+q\rvert\right)$	
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan\frac{x}{a}$	
$\frac{x}{x^2+a^2}$	$\frac{1}{2}\ln(x^2+a^2)$	
$\sqrt{x^2+a^2}$	$\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})$	
$\sqrt{x^2-a^2}$	$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\lvert x+\sqrt{x^2-a^2}\rvert$	
$\sqrt{a^2-x^2}$	$\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$	
$\frac{1}{\sqrt{x^2+a^2}}$	$\ln(x+\sqrt{x^2+a^2})$	
$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\lvert x+\sqrt{x^2-a^2}\rvert$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\frac{x}{a}$	
$\frac{1}{(x^2+a^2)^{3/2}}$	$\frac{x}{a^2\sqrt{x^2+a^2}}$	
$\sin ax$	$-\frac{1}{a}\cos ax$	

$\sin^2 ax$	$\frac{1}{2a}(ax - \sin ax \cos ax)$
$\frac{1}{\sin ax}$	$\frac{1}{a} \ln \left\lvert \tan \frac{ax}{2} \right\rvert$
$x \sin ax$	$\frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$
$x^2 \sin ax$	$\frac{2x}{a^2} \sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right) \cos ax$
$\cos ax$	$\frac{1}{a} \sin ax$
$\cos^2 ax$	$\frac{1}{2a}(ax + \sin ax \cos ax)$
$\frac{1}{\cos ax}$	$\frac{1}{a} \ln \left\lvert \tan \left(\frac{\pi}{4} + \frac{ax}{2}\right) \right\rvert$
$x \cos ax$	$\frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$
$x^2 \cos ax$	$\frac{2x}{a^2} \cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right) \sin ax$
$\frac{1}{\cos^2 ax}$	$\frac{1}{a} \tan ax$
$\frac{1}{\sin^2 ax}$	$-\frac{1}{a} \cot ax$
$\tan ax$	$-\frac{1}{a} \ln \lvert \cos ax \rvert$
$\ln ax$	$x \ln ax - x$
e^{ax}	$\frac{e^{ax}}{a}$
xe^{ax}	$\frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$
$x^2 e^{ax}$	$\frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right)$
$x^n e^{ax}$	$\frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx$ $= \frac{e^{ax}}{a}\left(x^n - \frac{n x^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \ldots \frac{(-1)^n n!}{a^n}\right)$ if n = positive integer

Definite integrals ($a > 0$)

$$\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{a^2 \pi}{4}$$

$$\int_0^\infty e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{1}{a^{n+1}} \Gamma(n+1)$$

where Γ is the gamma-function $\Gamma(m+1) = m\,\Gamma(m)$, $\Gamma(1) = 1$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\int_0^\infty x^m e^{-ax^2}\, dx = I_m = \frac{1}{2a^{(m+1)/2}} \Gamma[(m+1)/2] \qquad I_m = \left(\frac{m-1}{2a}\right) I_{m-2}$$

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = 2\int_0^\infty \frac{x}{e^x + 1}\, dx = \frac{\pi^2}{6}$$

$$\int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{\pi^4}{15}$$

$$\int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}$$

$$\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}$$

$$\int_0^\pi \sin mx \sin nx\, dx = \begin{cases} 0 & m, n \text{ integers; } m \neq n \\ \pi/2 & m, n \text{ integers; } m = n \end{cases}$$

$$\int_0^\pi \cos mx \cos nx\, dx = \begin{cases} 0 & m, n \text{ integers; } m \neq n \\ \pi/2 & m, n \text{ integers; } m = n \end{cases}$$

$$\int_0^\pi \sin mx \cos nx\, dx = \begin{cases} 0 & m, n \text{ integers; } m + n \text{ even} \\ 2m/(m^2 - n^2) & m, n \text{ integers; } m + n \text{ odd} \end{cases}$$

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \frac{(n-1)!!}{n!!} & n \text{ odd} \\ \frac{(n-1)!!}{n!} \cdot \frac{\pi}{2} & n \text{ even} \end{cases}$$

$$\int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = \frac{\pi a}{2}$$

($n!! = n\,(n-2)\,(n-4)\cdot\ldots\cdot n_1$ where $n_1 = 1$ or 2)

$$\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}$$

Divergence theorem (Gauss' theorem, Green's theorem)

$$\int_V \nabla\cdot \boldsymbol{A}\,dV = \int_S \boldsymbol{A}\cdot d\boldsymbol{S}$$

where S is a closed surface bounding a region of volume V and $d\boldsymbol{S} = \boldsymbol{e}_N\,dS$ where $\boldsymbol{e}_N$ is the positive (outward) normal.

Stokes' theorem

$$\oint_C \boldsymbol{A}\cdot d\boldsymbol{r} = \int_S (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{S}$$

where S is a two-sided surface bounded by a closed non-intersecting curve C (simple closed curve).

Green's theorem in the plane

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where R is the area bounded by the simple closed curve C.

Miscellaneous theorems

$$\int_V \nabla\times\boldsymbol{A}\,dV = \int_S d\boldsymbol{S}\times\boldsymbol{A}$$

$$\int_C f\,d\boldsymbol{r} = \int_S d\boldsymbol{S}\times\nabla f$$

7 Taylor Series

Taylor formula

$$f(x) = f(a) + \frac{1}{1!}f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \ldots + \frac{1}{n!}f^{(n)}(\xi)(x-a)^n$$

where $\xi \in [a, x]$

Special cases for $a = 0$ (Maclaurin series)

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$

$$\sin x = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots \qquad |x| < \frac{\pi}{2}$$

$$\cot x = \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 + \ldots \qquad 0 < |x| < \pi$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \qquad |x| < 1$$

$$(1+x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \ldots \qquad |x| < 1$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \ldots$$

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots \qquad |x| < 1$

$\arcsin x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots \qquad |x| < 1$

$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots$

$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots$

8 Special Polynomials and Associated Functions

(See also Sec. M – 12)

Hermite polynomials

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 + 12$

...

Legendre polynomials

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

...

$P_0(\cos\theta) = 1$

$P_1(\cos\theta) = \cos\theta$

$P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)$

$P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

...

Associated Legendre functions

$P_1^1(x) = (1 - x^2)^{1/2}$

$P_2^1(x) = 3x(1 - x^2)^{1/2}$

$P_2^2(x) = 3(1 - x^2)$

...

$P_3^1(x) = \frac{3}{2}(5x^2 - 1)(1 - x^2)^{1/2}$

$P_3^2(x) = 15x(1 - x^2)$

$P_3^3(x) = 15(1 - x^2)^{3/2}$

...

Laguerre polynomials

$L_0(x) = 1$

$L_1(x) = -x + 1$

$L_2(x) = x^2 - 4x + 2$

Associated Laguerre polynomials

$L_1^1(x) = -1$

$L_2^1(x) = 2x - 4$

$L_2^2(x) = 2$

...

$L_3^1(x) = -3x^2 + 18x - 18$

$L_3^2(x) = -6x + 18$

$L_3^3(x) = -6$

...

Chebyshev polynomials

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_5(x) = 16x^5 - 20x^3 + 5x$

...

$1 = T_0$

$x = T_1$

$x^2 = \frac{1}{2}(T_0 + T_2)$

$x^3 = \frac{1}{4}(3T_1 + T_3)$

$x^4 = \frac{1}{8}(3T_0 + 4T_2 + T_4)$

$x^5 = \frac{1}{16}(10T_1 + 5T_3 + T_5)$

...

Bessel functions of the first kind of order 0 and 1

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \dots$$

9 Vector Analysis

Scalar product

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}| \cdot |\boldsymbol{B}| \cos(\boldsymbol{A}, \boldsymbol{B}) = A_x B_x + A_y B_y + A_z B_z$$

$$|\boldsymbol{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Vector product

$$|\boldsymbol{A} \times \boldsymbol{B}| = |\boldsymbol{A}|\,|\boldsymbol{B}| \sin(\boldsymbol{A}, \boldsymbol{B})$$

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} =$$

$$= (A_y B_z - A_z B_y)\,\hat{\boldsymbol{x}} + (A_z B_x - A_x B_z)\,\hat{\boldsymbol{y}} + (A_x B_y - A_y B_x)\,\hat{\boldsymbol{z}}$$

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$$
$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C})$$
$$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C})$$
$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

Miscellaneous formulae involving $\nabla = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{(Laplace operator)}$$
$$\nabla \times (\nabla f) = \boldsymbol{0}$$
$$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$$
$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$
$$\nabla \cdot (f\boldsymbol{A}) = (\nabla f) \cdot \boldsymbol{A} + f(\nabla \cdot \boldsymbol{A})$$
$$\nabla \times (f\boldsymbol{A}) = f(\nabla \times \boldsymbol{A}) + (\nabla f) \times \boldsymbol{A}$$
$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$$
$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A})$$
$$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$$

$$\operatorname{grad} f = \nabla f$$
$$\operatorname{div} \boldsymbol{A} = \nabla \cdot \boldsymbol{A}$$
$$\operatorname{curl} \boldsymbol{A} = \operatorname{rot} \boldsymbol{A} = \nabla \times \boldsymbol{A}$$

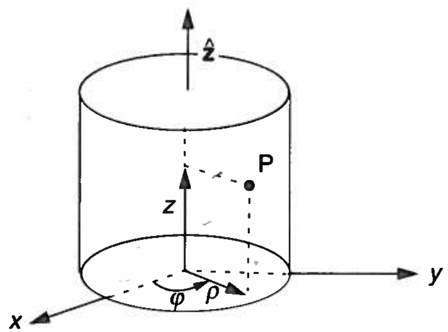
10 Special Coordinate Systems

Cylindrical coordinates (ρ, φ, z)

$x = \rho \cos \varphi$

$y = \rho \sin \varphi$

$z = z$



$$dV = \rho \, d\rho \, d\varphi \, dz$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \varphi} A_\varphi + \frac{\partial}{\partial z} A_z$$

$$\nabla f = \frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} \hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

(See also Section F–1.5)

$$\nabla \times \boldsymbol{A} = \frac{1}{\rho} \left(\frac{\partial}{\partial \varphi} A_z - \frac{\partial}{\partial z} (\rho A_\varphi) \right) \hat{\boldsymbol{\rho}} + \left(\frac{\partial}{\partial z} A_\rho - \frac{\partial A_z}{\partial \rho} \right) \hat{\boldsymbol{\varphi}} + \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi} \right) \hat{\mathbf{z}}$$

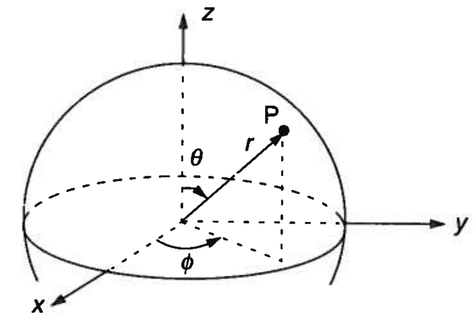
$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{A} \cdot d\boldsymbol{r} = \int_{P_0}^{P_1} (A_\rho \, d\rho + A_\varphi \rho \, d\varphi + A_z \, dz)$$

Spherical coordinates (r, θ, ϕ)

$x = r \sin \theta \cos \phi$

$y = r \sin \theta \sin \phi$

$z = r \cos \theta$



$$dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin\theta \, A_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} A_\phi$$

$$\nabla f = \frac{\partial f}{\partial r} \hat{\boldsymbol{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} =$$

$$= \frac{1}{r} \frac{\partial^2}{\partial r^2} (rf) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

(See also Section F–1.5)

$$\nabla \times \boldsymbol{A} = \frac{1}{r^2 \sin \theta} \left(\left(\frac{\partial}{\partial \theta} (r \sin\theta \, A_\phi) \right) - \frac{\partial}{\partial \phi} (r A_\theta) \right) \hat{\boldsymbol{r}} +$$

$$+ \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \phi} A_r - \frac{\partial}{\partial r} (r \sin\theta \, A_\phi) \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial}{\partial \theta} A_r \right) \hat{\boldsymbol{\phi}}$$

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{A} \cdot d\boldsymbol{r} = \int_{P_0}^{P_1} (A_r \, dr + A_\theta r \, d\theta + A_\phi r \sin \theta \, d\phi)$$

Any orthogonal curvelinear coordinates u_1, u_2, u_3

$$A = A_1\hat{u}_1 + A_2\hat{u}_2 + A_3\hat{u}_3$$

$$\operatorname{div} A = \nabla\cdot A = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

$$\operatorname{grad} f = \nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{u}_3$$

$$\Delta f = \nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

$$\operatorname{curl} A = \nabla\times A = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3A_3) - \frac{\partial}{\partial u_3}(h_2A_2)\right]\hat{u}_1$$

$$+ \frac{1}{h_1h_3}\left[\frac{\partial}{\partial u_3}(h_1A_1) - \frac{\partial}{\partial u_1}(h_3A_3)\right]\hat{u}_2 + \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\hat{u}_3$$

For cylindrical coordinates

$$h_1^2 = 1, h_2^2 = \rho^2, h_3^2 = 1$$

For spherical coordinates

$$h_1^2 = 1, h_2^2 = r^2, h_3^2 = r^2\sin^2\theta$$

11 Laplace Transforms

Definition

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,\ s > \alpha \text{ where } \alpha \text{ is some constant}$$

Transform	**Original function**
$F(s)$	$f(t)$
$F(s+a)$	$e^{-at} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a); & t-a>0 \\ 0 & ; t-a<0 \end{cases}$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(as)\quad (a>0)$	$\frac{1}{a} f\left(\frac{t}{a}\right)$
$\frac{d^nF(s)}{ds^n}$	$(-t)^n f(t)$
$\int_s^\infty F(\sigma)\,d\sigma$	$\frac{f(t)}{t}$
$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_1(\sigma)F_2(s-\sigma)\,d\sigma$	$f_1(t)f_2(t)$
$F_1(s)\,F_2(s)$	$\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau = \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau$
$s\,F(s) - f(0)$	$f'(t)$
$s^2\,F(s) - [s\,f(0) + f'(0)]$	$f''(t)$
$s^n\,F(s) - [s^{n-1} f(0) + \ldots + f^{(n-1)}(0)]$	$f^{(n)}(t)$
$\frac{1}{s}F(s) + \frac{1}{s}\left[\int_0^t f(\tau)\,d\tau\right]_{t=+0}$	$\int_0^t f(\tau)\,d\tau$