Coefficients of viscosity, self-diffusion and thermal conductivity according to the diffusion approximation

(the formulae hold only for gases and give only the order of η, D and λ over a limited temperature range)

$$\eta \approx \frac{1}{\pi d^2}\sqrt{\frac{m\,k\,T}{\pi}} \approx \frac{D\,N\,m}{V}$$

m and d are the molecular mass and linear "diameter", respectively

$$D \approx \frac{1}{3}\,\langle v\rangle\,\ell$$

$\langle v\rangle$ for an ideal gas is given in Section 2.1

ℓ = mean free path

$$\lambda \approx \frac{3D\,k\,N}{2V} \approx \frac{3k\,\eta}{2m}$$

3 Electromagnetic Theory

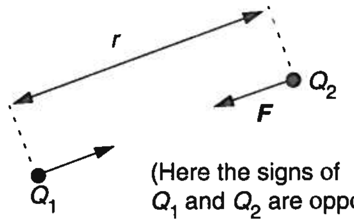
Quantity	Symbol	SI Unit
Position vector	$\boldsymbol{r}$	m
Frequency	f	$\mathrm{Hz = s^{-1}}$
Angular frequency	ω	rad/s
Phase difference	φ	rad
Length	ℓ, a, r	m
Area	A, S	$\mathrm{m^2}$
Velocity	$\boldsymbol{v}\,(v)$	m/s
Force	$\boldsymbol{F}\,(F)$	$\mathrm{N = kg\,m/s^2}$
Electric current	I	A
Electric current density	$\boldsymbol{j}\,(j)$	$\mathrm{A/m^2}$
Electric charge	Q	C = As
Charge density	ρ	$\mathrm{C/m^3}$
Surface charge	σ	$\mathrm{C/m^2}$
Electric potential and voltage*	V	V = Nm/As
Energy	W	J = Nm = VC
Energy density	w	$\mathrm{J/m^3}$
Power	P	W = VA = J/s
Electric field	$\boldsymbol{E}\,(E)$	V/m = N/C
Polarization	$\boldsymbol{P}$	$\mathrm{C/m^2}$
Electric displacement field	$\boldsymbol{D}$	$\mathrm{C/m^2}$
Relative permittivity	ε_r	
Resistance	R	Ω = V/A
Complex impendance	Z	Ω
Reactance	X	Ω
Capacitance	C	F = C/V = s/Ω
Inductance	L	H = Vs/A = Ωs
Electric conductivity	σ	A/Vm
Magnetic flux density	$\boldsymbol{B}\,(B)$	$\mathrm{T = Vs/m^2}$
Magnetic flux	Φ	$\mathrm{Wb = T\,m^2 = Vs}$
Magnetizing field	$\boldsymbol{H}\,(H)$	A/m
Magnetization	$\boldsymbol{M}$	A/m
Number of turns	N	
Relative permeability	μ_r	
Magnetic vector potential	$\boldsymbol{A}$	T m

* Recommended symbols are V or φ for electric potential and U or V for potential difference. We have adopted the symbol V for both quantities, in accordance with British and US practice.

3.1 The Coulomb Field

Coulomb's law

$$F = \frac{Q_1\,Q_2}{4\pi\,\varepsilon_0\,r^2}$$



(Here the signs of Q_1 and Q_2 are opposite)

Field intensity and potential

$$\boldsymbol{E} = \boldsymbol{F}/Q \qquad \boldsymbol{E} = \frac{1}{4\pi\,\varepsilon_0}\int \frac{\rho}{r^2}\frac{\boldsymbol{r}}{r}\,\mathrm{d}v \qquad v = \text{volume}$$

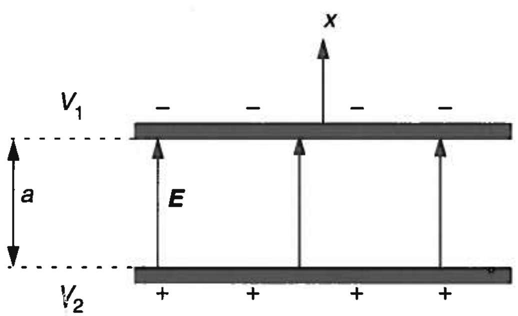
$\boldsymbol{F}$ = force on a charge Q in the electric field $\boldsymbol{E}$

$$\boldsymbol{E} = -\,\mathrm{grad}\;V \qquad V_A - V_B = -\int_B^A \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r}$$

$$V = \frac{1}{4\pi\,\varepsilon_0}\int \frac{\rho}{r}\,\mathrm{d}v$$

Uniform field

$$\boldsymbol{E} = \frac{V_2 - V_1}{a}\,\hat{\boldsymbol{x}}$$



Field around an electric point charge and outside a spherical charge distribution

$$E = \frac{1}{4\pi\,\varepsilon_0}\frac{Q}{r^2} \qquad \boldsymbol{E} = \frac{Q}{4\pi\,\varepsilon_0\,r^3}\,\boldsymbol{r}$$

$$V = \frac{1}{4\pi\,\varepsilon_0}\frac{Q}{r}$$

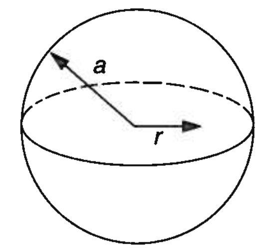
r = distance from centre of charge distribution

Field inside a spherical, uniform charge distribution

$$E_r = \frac{\rho\,r}{3\,\varepsilon_0} = \frac{Q\,r}{4\pi\,\varepsilon_0\,a^3}$$

$$V = \frac{Q}{8\pi\,\varepsilon_0\,a^3}\,(3a^2 - r^2)$$

Q = total charge of sphere



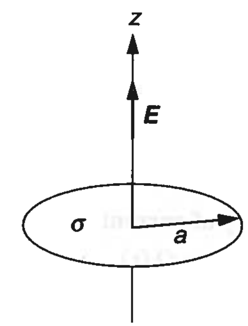
Field on axis of circular, uniform charge distribution

$$E_z = \frac{\sigma\,z}{2\,\varepsilon_0}\left(\frac{1}{|z|} - \frac{1}{\sqrt{z^2+a^2}}\right)$$

$$V = \frac{\sigma}{2\,\varepsilon_0}\left(\sqrt{z^2+a^2} - |z|\right)$$

Surface charge density

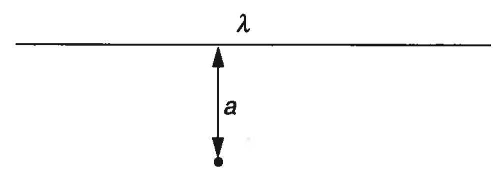
$$\sigma = \frac{Q}{\pi\,a^2} \qquad Q = \text{total disc charge}$$



Field from an infinitely long conductor

$$E = \frac{\lambda}{2\pi\,\varepsilon_0\,a}$$

$$V = -\,\frac{\lambda}{2\pi\,\varepsilon_0}\,\ln\frac{a}{a_0}$$



λ = linear charge density
a_0 = distance from conductor where $V = 0$

Poisson's equation

$$\nabla^2 V = -\,\rho/\varepsilon_0$$

Gauss' law of flux of a field vector

$$\Phi_E = \oint_S \boldsymbol{E}\,\mathrm{d}\boldsymbol{S} = Q/\varepsilon_0$$

Q = total charge within the closed surface S.

Alternative formulation of Gauss' law in polarization-free space

$$\operatorname{div} \boldsymbol{E} = \rho / \varepsilon_0$$

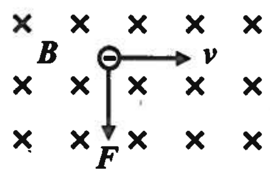
Irrotation of Coulomb field

$$\operatorname{curl} \boldsymbol{E} = \boldsymbol{0}$$

3.2 Motion of Charged Particles

Lorentz force

$$\boldsymbol{F} = Q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$



Direction of force $\boldsymbol{F}$ on a moving negatively charged particle in a magnetic field.

Electric current

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

Density of current

$$\boldsymbol{j} = n\,Q\,\langle \boldsymbol{v} \rangle \qquad j = I / A$$

n = number of charges Q per unit volume
$\langle \boldsymbol{v} \rangle$ = drift velocity of charge carriers

Total energy of one particle moving in an electric field (non-relativistic case)

$$W_{\text{tot}} = \tfrac{1}{2} m\, v^2 + Q\, V$$

m = particle mass

Drift velocity of charged particle acted on by a magnetic field and a non-magnetic force

$\boldsymbol{B}$ = magnetic flux density
$\boldsymbol{F}$ = non-magnetic force
q = particle charge
m = particle mass
$\boldsymbol{u}$ = drift velocity
$\boldsymbol{v}$ = particle velocity

Indices ⊥ and // mean "component perpendicular to magnetic field" and "component parallel to magnetic field", respectively.

$$\boldsymbol{u}_\perp = -\frac{1}{qB^2}\,\boldsymbol{B} \times \left(\boldsymbol{F} - \frac{1}{B}\frac{m v_\perp^2}{2} \operatorname{grad} B - m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \right)$$

$$\text{For } \boldsymbol{u}_{//}: \boldsymbol{B} \cdot \left(\boldsymbol{F} - \frac{1}{B}\frac{m v_\perp^2}{2} \operatorname{grad} B - m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \right) = 0$$

Special cases

1. $\boldsymbol{F} = q\boldsymbol{E}$; $\operatorname{grad} B \equiv \boldsymbol{0}$, $\dfrac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{0}$

$$\boldsymbol{u}_\perp = -\frac{1}{B^2}\,\boldsymbol{B} \times \boldsymbol{E} \quad \text{(indep. of } q, \boldsymbol{v}\text{)}$$

2. $\boldsymbol{F} = \dfrac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{0}$

$$\boldsymbol{u}_\perp = \frac{m v_\perp^2}{2qB^3}\,\boldsymbol{B} \times \operatorname{grad} B$$

3. $\dfrac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}$ dominated by centrifugal acceleration

$$\boldsymbol{u}_\perp = -\frac{1}{qB^2}\,\boldsymbol{B} \times \left[\boldsymbol{F} - \frac{m}{B}\left(\frac{v_\perp^2}{2} + v_{//}^2 \right) \operatorname{grad} B \right]$$

Generalized momentum

$$\boldsymbol{p} = m\,\boldsymbol{v} + Q\,\boldsymbol{A}$$

Generalized potential of Lorentz force

$$V_{\text{gen}} = Q(V - \boldsymbol{v} \cdot \boldsymbol{A})$$

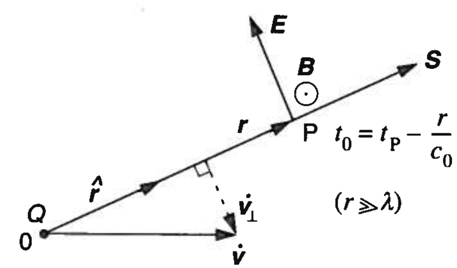
Radiation at a point P from an accelerated charge ($v \ll c_0$)

$$\boldsymbol{E}(\boldsymbol{r}, t_{\mathrm{P}}) = -\frac{Q}{4\pi\varepsilon_0 c_0^2 r}\,\dot{\boldsymbol{v}}_\perp(t_0)$$

$$\boldsymbol{B} = \frac{1}{c_0}\,(\hat{\boldsymbol{r}} \times \boldsymbol{E})$$

Poynting's vector

$$\boldsymbol{S} = \frac{1}{\mu_0}\,\boldsymbol{E} \times \boldsymbol{B}$$



See also Sec. F–5.3.

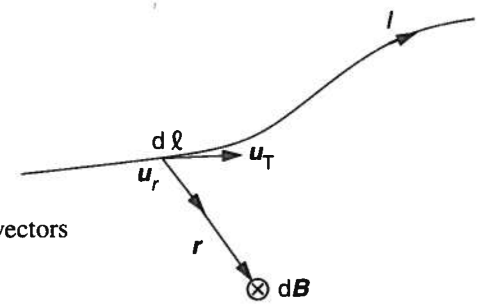
3.3 The Quasi-Stationary *B*-field

Ampere-Laplace law of the magnetic field from a closed loop

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint_L \frac{\boldsymbol{u}_T \times \boldsymbol{u}_r}{r^2} \, d\ell$$

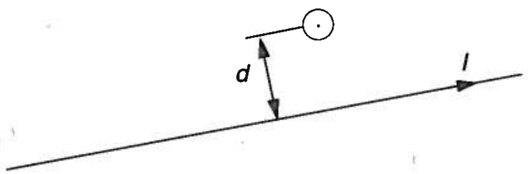
$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \frac{\boldsymbol{u}_T \times \boldsymbol{u}_r}{r^2} \, d\ell$$

$\boldsymbol{u}_T$ and $\boldsymbol{u}_r$ = unit vectors



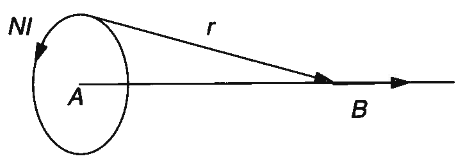
Example 1. The field at distance *d* from a long, straight conductor (Biot-Savart formula)

$$B = \frac{\mu_0}{4\pi} \frac{2I}{d}$$



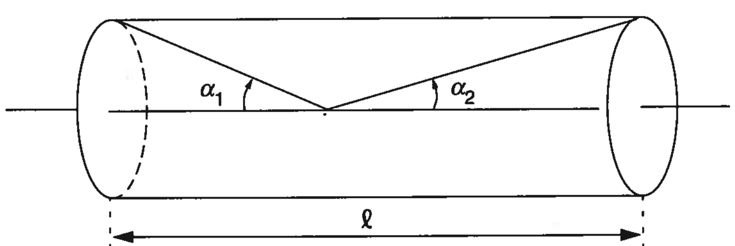
Example 2. Field on axis of circular loop (short solenoid)

$$B = \frac{\mu_0}{4\pi} \frac{2NIA}{r^3}$$



Example 3. Field on axis of thin solenoid

$$B = \mu_0 \frac{NI}{2\ell} (\cos\alpha_1 + \cos\alpha_2)$$



In particular, inside a long solenoid or a toroidal solenoid

$$B = \mu_0 \frac{NI}{\ell}$$

Exempel 4. Field from a charged particle in uniform motion

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \frac{Q}{r^2} \boldsymbol{v} \times \boldsymbol{u}_r$$

$$\boldsymbol{B} = \mu_0 \, \varepsilon_0 \, \boldsymbol{v} \times \boldsymbol{E}$$

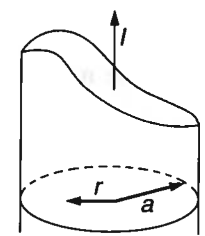
Ampere's law (law of circulation)

$\oint_L \boldsymbol{B} \, d\boldsymbol{r} = \mu_0 I$ $\qquad$ $\oint_L \boldsymbol{H} \, d\boldsymbol{r} = I$

curl $\boldsymbol{B} = \mu_0 \boldsymbol{j}$ $\qquad$ I = current inside the closed loop L

In particular, inside a conductor of uniform current density

$$B = \frac{\mu_0}{4\pi} \frac{2r}{a^2} I$$



Gauss' law of the magnetic flux

$\Phi_B = \oint_S \boldsymbol{B} \, d\boldsymbol{S} = 0$ $\qquad$ div $\boldsymbol{B} = 0$

Magnetic vector potential *A*

$\boldsymbol{B} = \text{curl}\,\boldsymbol{A}$ $\qquad$ $\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}}{r} \, dV$

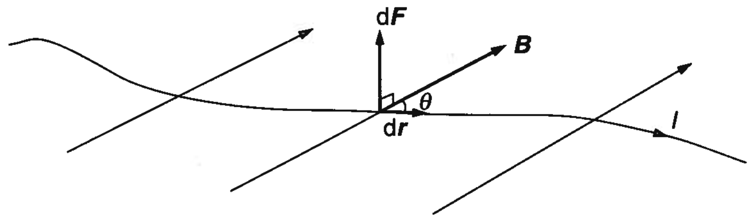
In particular, for a uniform ***B***-field

$$\boldsymbol{A} = \tfrac{1}{2} \, \boldsymbol{B} \times \boldsymbol{r}$$

Magnetic force on an electric conductor

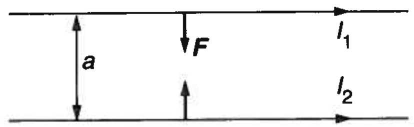
$dF = B\,I \sin\theta\, dr$

$d\boldsymbol{F} = I\, d\boldsymbol{r} \times \boldsymbol{B}$



In particular the mutual force on parallel conductors, each of length ℓ

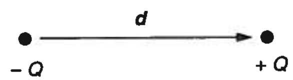
$$F = \frac{\mu_0}{4\pi}\,\frac{2\,\ell}{a}\, I_1 I_2$$



3.4 Electric and Magnetic Dipoles

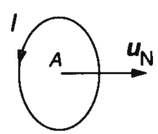
Electric dipole moment for charges + Q and – Q at a distance d from each other

$$\boldsymbol{p} = Q\,\boldsymbol{d}$$



Magnetic dipole moment of a small current loop

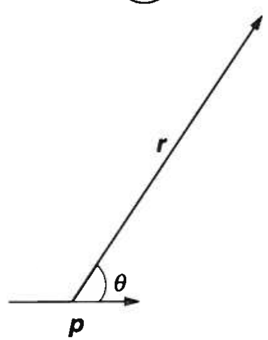
$\boldsymbol{m} = I\,A\,\boldsymbol{u}_N$ $\quad \boldsymbol{u}_N$ = unit vector



Potential far from an electric dipole

$$V = \frac{1}{4\pi\,\varepsilon_0}\,\frac{p\cos\theta}{r^2}$$

$$V = \frac{1}{4\pi\,\varepsilon_0}\,\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}$$



Electric field

$$\boldsymbol{E} = -\,\mathrm{grad}\, V$$

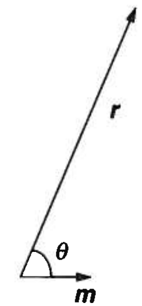
In particular, in spherical polar coordinates

$$E_r = -\frac{\partial V}{\partial r} \qquad E_\theta = -\frac{1}{r}\,\frac{\partial V}{\partial \theta}$$

Field far from a magnetic dipole

$$B_r = \frac{\mu_0}{4\pi}\,\frac{2m\cos\theta}{r^3}$$

$$B_\theta = \frac{\mu_0}{4\pi}\,\frac{m\sin\theta}{r^3}$$



Vector potential of a magnetic dipole

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\,\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}$$

Torque on dipole in external field

$\tau_e = p\,E\sin\alpha \qquad \tau_m = m\,B\sin\alpha$

$\boldsymbol{\tau}_e = \boldsymbol{p}\times\boldsymbol{E} \qquad \boldsymbol{\tau}_m = \boldsymbol{m}\times\boldsymbol{B}$

Potential energy of a dipole in external field

$W_e = -\boldsymbol{p}\cdot\boldsymbol{E} = -\,p\,E\cos\alpha$

$W_m = -\,\boldsymbol{m}\cdot\boldsymbol{B} = -\,m\,B\cos\alpha$

Force on dipole in external field

$\boldsymbol{F} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E} \qquad \boldsymbol{F} = (\boldsymbol{m}\cdot\nabla)\,\boldsymbol{B} + \boldsymbol{m}\times(\nabla\times\boldsymbol{B})$

In particular, if dipole is directed along x-axis, and if curl $\boldsymbol{B} = \boldsymbol{0}$.
(Force directed towards stronger field.)

$$\boldsymbol{F} = p_x\,\frac{\partial}{\partial x}\boldsymbol{E} \qquad \boldsymbol{F} = m_x\,\frac{\partial}{\partial x}\boldsymbol{B}$$

Electric dipole moment of an uncharged body with a charge distribution

$$p = \int_v r\,\rho\,\mathrm{d}v$$

v = volume of body

General definition of magnetic dipole moment

$$m = \tfrac{1}{2} \int r \times j\,\mathrm{d}v$$

3.5 Dielectric and Magnetic Media

Relations between E, D, and P

$$D = \varepsilon_0 E + P = \varepsilon_r \varepsilon_0 E$$

$$P = (\varepsilon_r - 1)\varepsilon_0 E = \chi_e \varepsilon_0 E$$

χ_e = electric susceptibility (sometimes χ_e is given such that $P = \chi_e E$)

P = dipole moment per unit volume = polarization

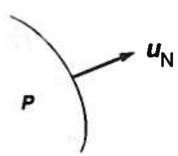
$\varepsilon_r \varepsilon_0$ = permittivity

D = electric displacement = electric flux density

Surface charge density from polarization

$$\sigma_p = P \cdot u_N$$

u_N = normal vector



Relations between B, H and M

$$B = \mu_0(H + M) = \mu_r \mu_0 H$$

$$M = (\mu_r - 1)H = \chi_m H$$

χ_m = magnetic susceptibility

M = dipole moment per volume = magnetization

Gauss' law of polarizable space

$$\mathrm{div}\, E = \frac{1}{\varepsilon_0}(\rho_f - \mathrm{div}\, P)$$

ρ_f = charge density other than that of polarization

$-\mathrm{div}\, P = \rho_p$ = charge density of polarization

Field vector components at interfaces

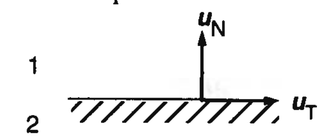
$$E_{1T} - E_{2T} = 0$$

$$D_{1N} - D_{2N} = \sigma_f$$

$$H_{1T} - H_{2T} = 0$$

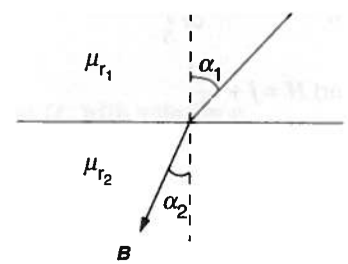
$$B_{1N} - B_{2N} = 0$$

σ_f = surface charge density other than that of polarization



In particular, the refraction of the B-field

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\mu_{r_1}}{\mu_{r_2}}$$

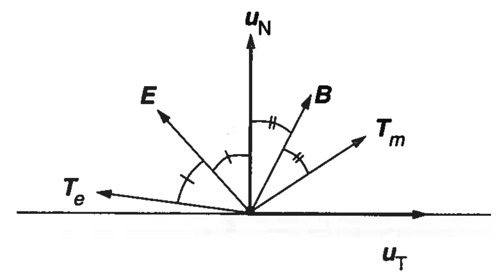


Maxwell stresses

$$T_e = \tfrac{1}{2}\,\varepsilon_0 E^2$$

$$T_m = \frac{1}{2\mu_0} B^2$$

$$T = T_e + T_m$$



Real force

$$\mathrm{d}F = T\,\mathrm{d}A$$

3.6 The Electromagnetic Field

Maxwell's equations

I $\oint_S \boldsymbol{D}\,\mathrm{d}\boldsymbol{S} = Q_f$ $\qquad Q_f$ = total free charge inside surface S

$\operatorname{div} \boldsymbol{D} = \rho_f$ $\qquad \rho_f$ = charge density other than that of polarization

$\operatorname{div} \boldsymbol{E} = \rho/\varepsilon_0$ (in polarization-free space)

II $\oint_S \boldsymbol{B}\,\mathrm{d}\boldsymbol{S} = 0$

$\operatorname{div} \boldsymbol{B} = 0$

III $\oint_L \boldsymbol{E}\,\mathrm{d}\boldsymbol{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B}\,\mathrm{d}\boldsymbol{S}$

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

IV $\oint_L \boldsymbol{H}\,\mathrm{d}\boldsymbol{r} = I + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{D}\,\mathrm{d}\boldsymbol{S}$ $\qquad \boldsymbol{D}$ = electric displacement = electric flux density

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\operatorname{curl} \boldsymbol{B} = \mu_0 \boldsymbol{j} + \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} \qquad (\text{when } \boldsymbol{P} = \boldsymbol{M} = \boldsymbol{0})$$

Law of charge conservation

$$\oint_S \boldsymbol{j}\,\mathrm{d}\boldsymbol{S} + \frac{\mathrm{d}}{\mathrm{d}t}\oint_S \boldsymbol{D}\,\mathrm{d}\boldsymbol{S} = 0$$

$$\operatorname{div} \boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0$$

Energy density in an electric and magnetic field respectively when ε_r and μ_r are constant

$$w_e = \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E} = \frac{1}{2}\varepsilon_r \varepsilon_0 E^2$$

$$w_m = \frac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B} = \frac{1}{2}\frac{1}{\mu_r \mu_0} B^2$$

In particular, total electrostatic energy of a chage distribution

$$W = \frac{1}{2}\int \boldsymbol{D}\cdot\boldsymbol{E}\,\mathrm{d}v = \frac{1}{2}\int \rho_f V\,\mathrm{d}v \qquad v = \text{volume}$$

Relation between ε_0 and μ_0

$$\varepsilon_0 \mu_0 = 1/c_0^2$$

Poynting vector (power flux density)

$$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H} = c_0^2 \varepsilon_0 \boldsymbol{E}\times\boldsymbol{B}$$

Effective penetration depth (skin effect)

$$\delta = 1/\sqrt{\pi \mu_r \mu_0 \sigma f} \qquad \sigma = \text{electric conductivity}$$

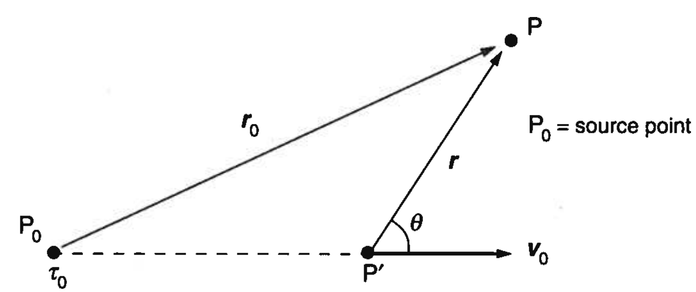
Vector field (at point P) from a point charge Q_1 with velocity v_0

$$\boldsymbol{E} = \frac{Q_1}{4\pi\varepsilon_0}\left(-\operatorname{grad}\frac{1}{r_1} - \frac{1}{c_0^2}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{v}_0}{r_1}\right)\right) = -\operatorname{grad} V - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \frac{1}{c_0}\frac{\boldsymbol{r}_0}{r_0}\times\boldsymbol{E} = \operatorname{curl}\boldsymbol{A}$$

Scalar field r_1

$$r_1 = r\sqrt{1 - \frac{v_0^2}{c_0^2}\sin^2\theta}$$



At time t the particle is found at P′

Example 1. Uniform motion

$$E = \frac{Q_1}{4\pi\,\varepsilon_0}\,\frac{r}{r_1^3}\left(1-\frac{v_0^2}{c_0^2}\right)$$

$$B = \frac{\mu_0\,Q_1}{4\pi}\,\frac{v_0\times r}{r_1^3}\left(1-\frac{v_0^2}{c_0^2}\right)$$

Example 2. Stationary or quasi-stationary particle

$$E = \frac{Q_1}{4\pi\,\varepsilon_0}\,\frac{r}{r^3}$$

$$B = \frac{\mu_0}{4\pi}\,Q_1\,\frac{v_0\times r}{r^3} = \frac{\mu_r\,\varepsilon_r}{c_0^2}\,v_0\times E$$

Relation between field time t and source time τ

$$\tau = t - r/c_0$$

3.7 Relativity and Electromagnetism

Transformation to a system S′, moving with a velocity v along the x-axis of the system S (both origins coincide at time $t = t' = 0$)

$$\rho' = \frac{\rho - v\,j_x/c_0^2}{\sqrt{1-\beta^2}} = \frac{\rho - v\cdot j/c_0^2}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c_0}$$

$$j'_{x'} = \frac{j_x - v\,\rho}{\sqrt{1-\beta^2}} = \frac{(j-\rho\,v)_x}{\sqrt{1-\beta^2}}$$

$$j'_{y',z'} = j_{y,z} = (j-\rho\,v)_{y,z}$$

$$E'_{x'} = E_x = (E + v\times B)_x$$

$$E'_{y',z'} = \frac{(E+v\times B)_{y,z}}{\sqrt{1-\beta^2}}$$

$$B'_{x'} = B_x = (B - v\times E/c_0^2)_x$$

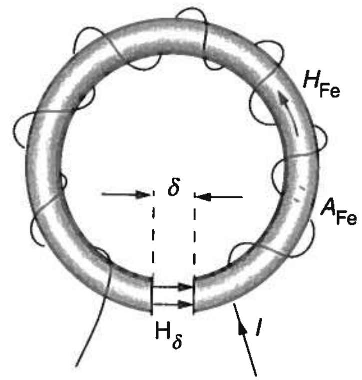
$$B'_{y',z'} = \frac{(B-v\times E/c_0^2)_{y,z}}{\sqrt{1-\beta^2}}$$

3.8 The Magnetic Circuit

Circulation law of a toroidal solenoid with small air gap δ

$$H_{Fe}\,\ell + H_\delta\,\delta = N\,I$$

ℓ = length of solenoid



Magnetizing field in the solenoid if the leak can be neglected

$$H_{Fe} = \frac{N\,I}{\ell+\delta} - \frac{\delta}{\ell+\delta}\,M_{Fe}$$

$$\frac{\delta}{\ell+\delta} = \text{demagnetization factor}$$

Magnetic flux

$$\Phi_B = B_{Fe}A_{Fe} = B_\delta A_\delta \qquad A = \text{cross-section area}$$

Example 1.

$$B_\delta = \mu_r\,\mu_0\,\frac{NI}{\ell - \delta + \mu_r\delta}$$

Example 2. Permanent magnet ($I = 0$)

$$H_{Fe}\,\ell + H_\delta\,\delta = 0$$

$$B_{Fe} = -H_{Fe}\,\frac{\mu_0\,\ell}{\delta}\,\frac{A_\delta}{A_{Fe}}$$

Reluctance and permeance of a magnetic "conductor" of length ℓ and area A

$$R_m = \ell/\mu_r\,\mu_0\,A$$

$$\Lambda = 1/R_m$$

"Ohm's law" of magnetic circuit

$$N\,I = R_m\,\Phi_B$$

Magnetic "pole mass" inside a closed surface S

$$Q_m = \oint_S H\,dS$$

Magnetic "pole density"

$$\rho_m = -\operatorname{div} M$$

3.9 Induction and Inductance

Law of induction (Faraday-Henry law)

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

ε = induced electromotive force, emf

$$\oint_L \boldsymbol{E}\, d\boldsymbol{r} = -\frac{d}{dt}\int_S \boldsymbol{B}\, d\boldsymbol{S}$$

Induced emf in a solenoid

$$\varepsilon = -N\frac{d\Phi_B}{dt}$$

Lenz's law

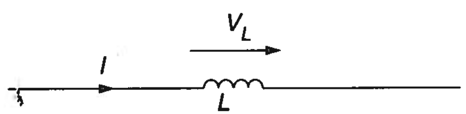
The direction of the induced current is such as to create a magnetic field which opposes the change of magnetic flux.

Definition of inductance L

$$N\Phi_B = LI$$

Potential drop over a pure inductor

$$V_L = L\frac{dI}{dt}$$



Inductance in a long straight or toroidal solenoid of length ℓ

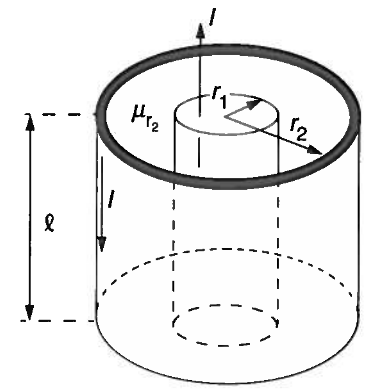
$$L = \mu_0 \frac{N^2 A}{\ell}$$

Inductance of a coaxial cable

$$L = \mu_{r_2}\mu_0 \frac{\ell}{2\pi}\ln\frac{r_2}{r_1} + \frac{\mu_{r_1}\mu_0 \ell}{8\pi}$$

μ_{r_1} = relative permeability of inner conductor

(The thickness of the outer conductor has been neglected.)



Equivalent inductance of inductors in series and in parallel

$$L_s = \Sigma_n L_n$$

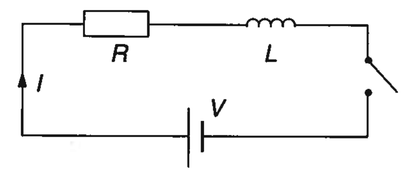
$$\frac{1}{L_p} = \Sigma_n \frac{1}{L_n}$$

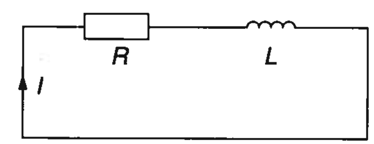
Energization and deenergization of an inductor

$$V = RI + L\frac{dI}{dt}$$

$$I = I_0\left(1 - e^{-\frac{R}{L}t}\right)$$

$$I = I_0\, e^{-\frac{R}{L}t}$$





Energy stored in solenoidal magnetic field

$$W_L = \frac{1}{2}LI^2$$

Energy density

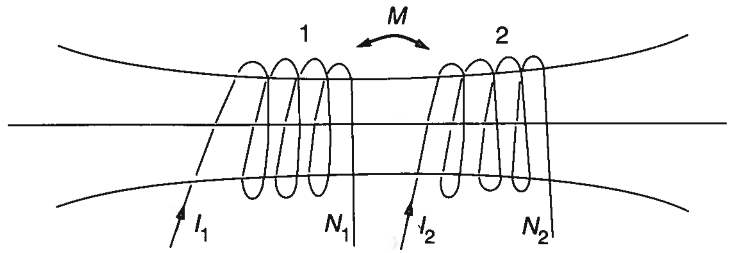
$$w_m = \frac{1}{2\mu_r\mu_0}B^2$$

Definition of mutual inductance M

$$N_2\,\Phi_{12} = M\,I_1$$

$$N_1\,\Phi_{21} = M\,I_2$$

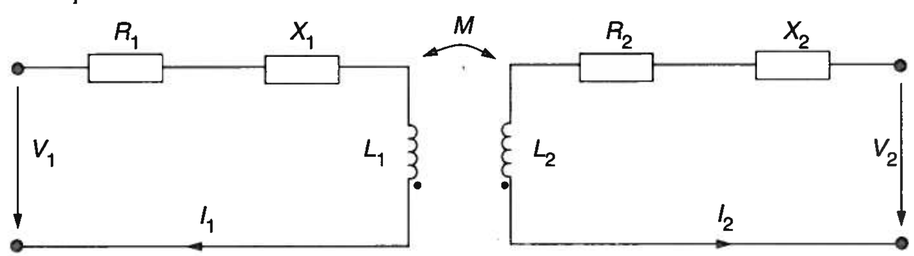
Φ_{12} = flux generated by coil 1 affecting coil 2 (in most cases $\approx \Phi_1$)



Coupling factor

$$k = \frac{M}{\sqrt{L_1\,L_2}} \qquad |k| < 1$$

Example. Air transformer

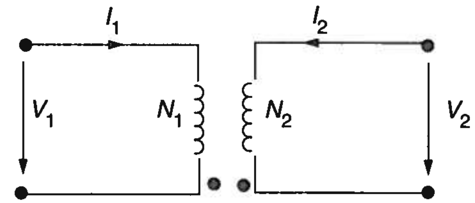


$Z_1 = R_1 + \mathrm{j}\,X_1 \qquad Z_2 = R_2 + \mathrm{j}\,X_2$

$V_1 = (Z_1 + \mathrm{j}\,\omega L_1)I_1 + \mathrm{j}\,\omega M\,I_2 \qquad V_2 = (Z_2 + \mathrm{j}\,\omega L_2)I_2 + \mathrm{j}\,\omega M\,I_1$

Ideal transformer

Equivalent circuit

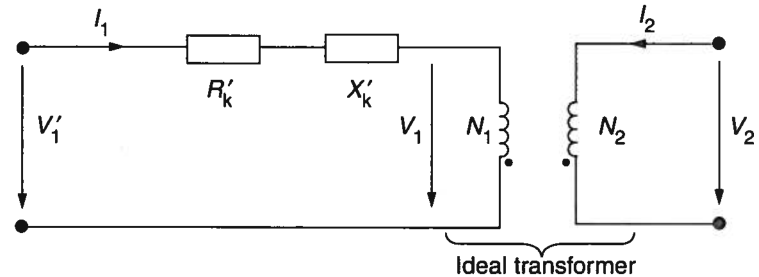


$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$

$$I_1\,N_1 = -I_2\,N_2$$

Real transformer

Equivalent circuit



$$R'_k = R_1 + \left(\frac{N_1}{N_2}\right)^2 R_2 \qquad X'_k = X_1 + \left(\frac{N_1}{N_2}\right)^2 X_2$$

Efficiency of transformer

$$\eta = \frac{P_2}{P_2 + P_0 + P_b}$$

$P_2 = V_2\,I_2 \cos\varphi_2 = x \cdot S_{2M} \cos\varphi_2$

x = load factor, i.e. $I_2 = x\,I_{2M}$

I_{2M} = rated current of secondary

$S_{2M} = V_{2M}\,I_{2M}$ = rated power of secondary

P_0 = no-load loss

$P_b = x^2\,P_{bM}$ = load loss

$P_{bM} = R'_k\,I^2_{1M}$ = maximum copper loss

I_{1M} = rated current of primary

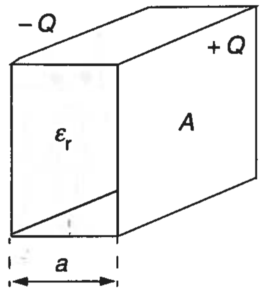
3.10 Capacitance

Definition of capacitance

$$C = Q/V$$

Parallel-plate capacitor

$$C = \varepsilon_r \, \varepsilon_0 \, \frac{A}{a}$$

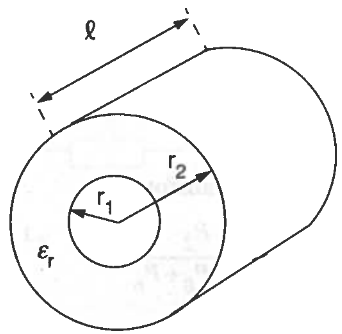


Mutual force on plates

$$F = \frac{\mathrm{d}W_C}{\mathrm{d}a} = \frac{Q^2}{2\varepsilon_r \, \varepsilon_0 \, A}$$

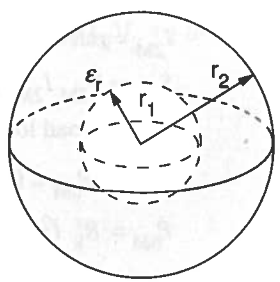
Cylinder capacitor

$$C = \varepsilon_r \, \varepsilon_0 \, \frac{2\pi \, \ell}{\ln(r_2/r_1)}$$



Spherical capacitor

$$C = \varepsilon_r \, \varepsilon_0 \, \frac{4\pi \, r_1 \, r_2}{r_2 - r_1}$$



Leakage conductance

$$\frac{1}{R} = \frac{\sigma \, C}{\varepsilon_r \, \varepsilon_0}$$ σ = conductivity of dielectric

Resultant capacitance in series and parallel arrangements

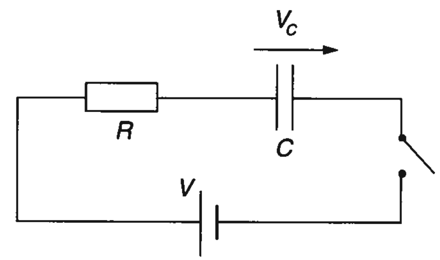
$$\frac{1}{C_s} = \Sigma_n \frac{1}{C_n}$$

$$C_p = \Sigma_n \, C_n$$

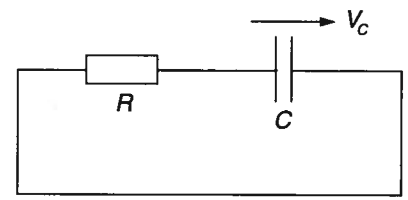
Charging and discharging of a capacitor through a resistor

$$V = V_C + R\,C\,\frac{\mathrm{d}\,V_C}{\mathrm{d}t}$$

$$V_C = V(1 - \mathrm{e}^{-t/RC})$$



$$V_C = V_0 \, \mathrm{e}^{-t/RC}$$



Energy of electric field in capacitor

$$W_C = \tfrac{1}{2}\,C\,V^2 = \tfrac{1}{2}\,Q^2/C = \tfrac{1}{2}\,Q\,V$$

Energy density

$$w_e = \tfrac{1}{2}\,\varepsilon_r\,\varepsilon_0\,E^2 = \tfrac{1}{2}\,D\,E$$

3.11 The Electric Circuit

Ohm's law

$V = R\,I$

$j = \sigma E$

Electric conductivity of a conductor

$\sigma = \dfrac{\ell}{R\,A}$

ℓ = length of conductor

A = cross-section area

Resistivity and conductance

$\rho = \dfrac{1}{\sigma} = \dfrac{R\,A}{\ell}$

$G = 1/R$

Temperature dependence of resistivity

$\rho = \rho_0(1 + \alpha(T - T_0))$

α = temperature coefficient

T = temperature

Equivalent resistance of resistors in series and in parallel

$R_s = \Sigma_n R_n$

$\dfrac{1}{R_p} = \Sigma_n \dfrac{1}{R_n}$

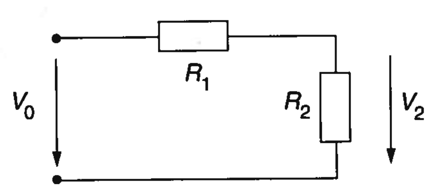
In particular, two parallel resistors

$R_p = \dfrac{R_1 R_2}{R_1 + R_2}$

The same rule applies for impedances Z in series and parallel.

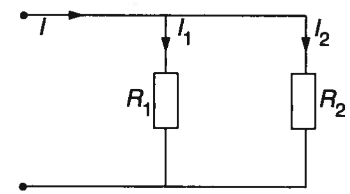
Voltage division

$V_2 = \dfrac{R_2}{R_1 + R_2} V_0$



Current division

$I_2 = \dfrac{R_1}{R_1 + R_2} I$



Kirchhoff's laws

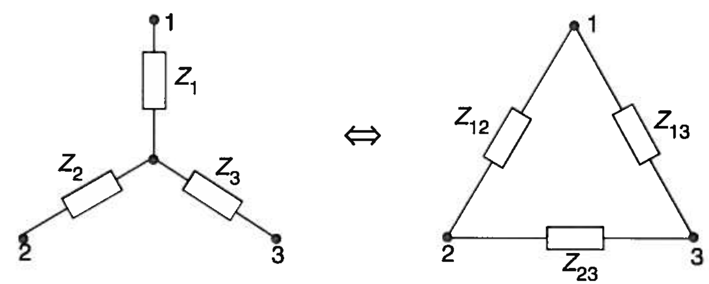
$\Sigma_i I_i = 0$ at a junction

$\Sigma_i V_i = 0$ round a closed path

Joule's law for thermal power

$P = V\,I = R\,I^2 = V^2/R$

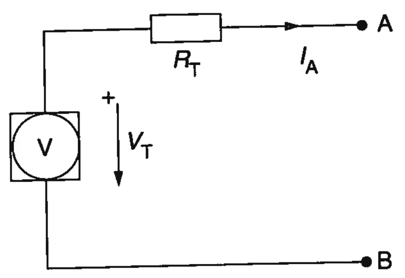
Transformation between Y and delta connections



$Z_1 = \dfrac{Z_{12} Z_{13}}{Z_{12} + Z_{13} + Z_{23}}$

$Z_{12} = Z_1 Z_2 \left(\dfrac{1}{Z_1} + \dfrac{1}{Z_2} + \dfrac{1}{Z_3}\right)$

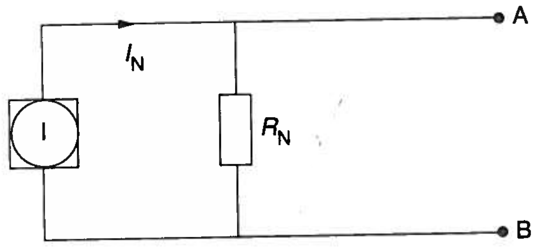
Thevenin equivalent circuit of an active two-terminal network



V_T = open-circuit potential difference between A and B ($I_A = 0$)

R_T = resistance between A and B when all voltage supplies are short-circuited and all current supplies are inactive.

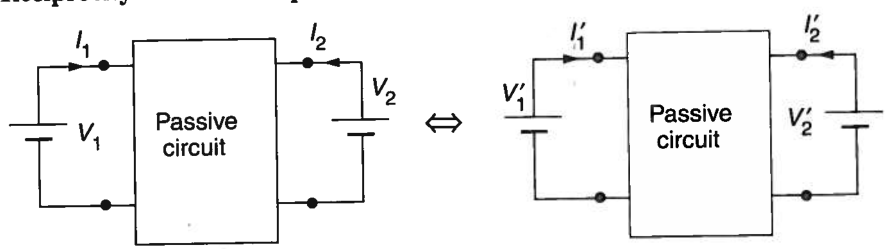
Norton equivalent of an active two-terminal network



I_N = current through A – B when output is short-circuited

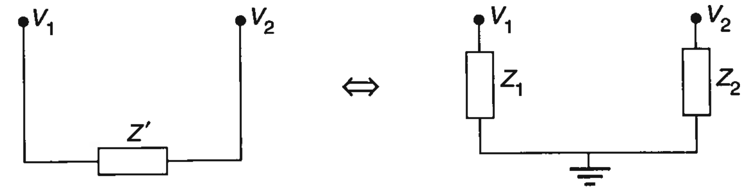
$R_N = R_T$

Reciprocity theorem of a passive circuit



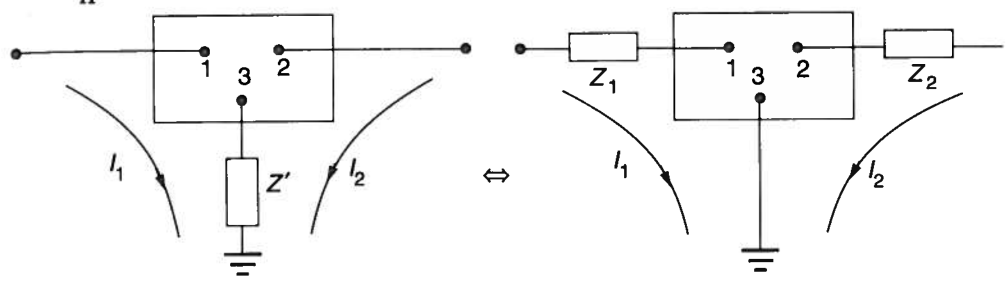
$$V_1 I_1' + V_2 I_2' = V_1' I_1 + V_2' I_2$$

I



$$Z_1 = \frac{Z'}{1-A_v} \qquad Z_2 = \frac{Z'}{1-1/A_v} \qquad A_v = \frac{V_2}{V_1}$$

II



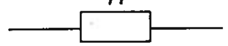
$$Z_1 = (1 - A_i)Z' \qquad Z_2 = \left(1 - \frac{1}{A_i}\right)Z' \qquad A_i = -\frac{I_2}{I_1}$$

3.12 Alternating (AC) Currents

Relations between frequency, angular frequency, and period T

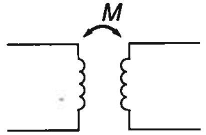
$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Impendance and phase difference of voltage in different (ideal) circuit elements relative to current

$Z = R$  $\varphi = 0°$

$Z = \mathrm{j}\,\omega L$  $\varphi = +90°$

$Z = -\mathrm{j}/\omega C$  $\varphi = -90°$

$Z = \mathrm{j}\,\omega M$  $\varphi = +90°$

M = mutual inductance as in Section 3.9

$\mathrm{j} = \sqrt{-1}$

Overall impendance

$$Z = R + \mathrm{j}\,X \qquad \varphi = \arctan\frac{X}{R}$$

Admittance

$Y = 1/Z = G + \mathrm{j}\,B$

G = conductance
B = susceptance

Effective value

$$I = \left(\frac{1}{T}\int_0^T i^2\,\mathrm{d}t\right)^{1/2} \qquad V = \left(\frac{1}{T}\int_0^T v^2\,\mathrm{d}t\right)^{1/2}$$

In particular,

$$i = \hat{\imath}\,\sin\omega t \;\Rightarrow\; I = \hat{\imath}/\sqrt{2}$$

$$v = \hat{v}\,\sin\omega t \;\Rightarrow\; V = \hat{v}/\sqrt{2}$$

Effective power

$P = V\,I\,\cos\varphi = R\,I^2$ $\qquad\cos\varphi$ = power factor

Reactive power

$$Q = V\,I\,\sin\varphi = X\,I^2$$

Apparent power

$$S = V\,I$$

General definition of quality factor

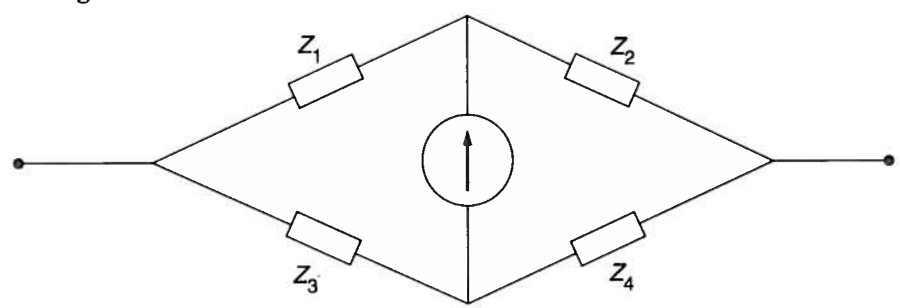
$$Q_0 = \omega_0\,\frac{\langle W_L\rangle + \langle W_C\rangle}{P} \quad \text{at resonance}$$

ω_0 = angular frequency at resonance

$\langle W_L\rangle$ and $\langle W_C\rangle$ are time mean values of stored energy in inductors and capacitors.

$$\langle W_L\rangle = \frac{1}{2}\,L\,I^2 \qquad \langle W_C\rangle = \frac{1}{2}\,C\,V_C^{\,2}$$

"Bridge"

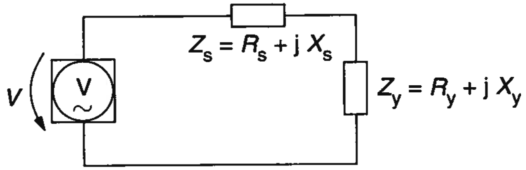


Condition of balance

$$Z_1\,Z_4 = Z_2\,Z_3$$

Power adjustment

$$P_y = \frac{R_y V^2}{(R_s+R_y)^2+(X_s+X_y)^2}$$



Condition for maximum power in Z_y

$$R_y = R_s \text{ and } X_y = -X_s \Rightarrow P_{max} = \frac{V^2}{4 R_s}$$

if R_y and X_y may be chosen independently

$$|Z_y| = |Z_s| \Rightarrow P_{max} = \frac{V^2 R_y}{|Z_s+Z_y|^2} = \frac{V^2 \cos \varphi_y}{2|Z_s|(1+\cos(\varphi_y-\varphi_s))}$$

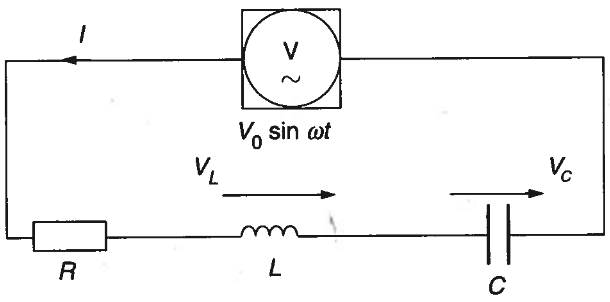
if only the magnitude of Z_y may be varied (e.g. a resistor).

$$\varphi_y = \arctan \frac{X_y}{R_y} \qquad \varphi_s = \arctan \frac{X_s}{R_s}$$

$$R_y = |Z_y| \cos \varphi_y \qquad R_s = |Z_s| \cos \varphi_s$$

3.13 Series and Parallel Circuits

The Series Circuit



$$L \frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C} I = \omega V_0 \cos \omega t$$

Stationary solution

$$I = I_0 \sin(\omega t - \varphi)$$

$$I_0 = \frac{V_0}{\sqrt{R^2+(\omega L - 1/\omega C)^2}}$$

$$\varphi = \arctan \frac{\omega L - 1/\omega C}{R}$$

Quality

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

Resonant frequency

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$$

Impedance

$$Z = R + j(\omega L - 1/\omega C) = R(1 + j\, a) \qquad j = \sqrt{-1}$$

Normalized impendance

$$a = Q_0 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

Half-power points ω_1 and ω_2 (when $a = \pm 1$)

$$\frac{\omega_{1,2}}{\omega_0} = \frac{\pm 1}{2 Q_0} + \sqrt{\frac{1}{4 Q_0^2} + 1}$$

$$Q_0 = \frac{\omega_0}{\omega_1 - \omega_2}$$

$$\omega_1 \omega_2 = \omega_0^2$$

Band with (in Hz)

$$B = \frac{\omega_1 - \omega_2}{2\pi} = \frac{\omega_0}{2\pi Q_0} = \frac{1}{2\pi}\frac{R}{L}$$

Formulae for $\omega \approx \omega_0$, $Q_0 \gg 1$

$$V_L \approx j\, Q_0 V$$

$$V_C \approx -j\, Q_0 V$$

$$Z \approx R(1 + j\, 2\delta Q_0)$$

$$\delta = \frac{\omega - \omega_0}{\omega_0} = \text{relative frequency difference}$$

Switching off a power source (Oscillating circuit)

I. $\omega_0 > \gamma$

$$\omega_e = \sqrt{\omega^2 - \gamma^2} \qquad \omega_0 = 1/\sqrt{LC}$$

$$I = I_0\,e^{-\gamma t}\sin(\omega_e t + \alpha) \qquad \gamma = R/2L$$

Damping ratio

$$K = e^{\gamma T} \qquad T = \frac{2\pi}{\omega_e}$$

Logarithmic decrement

$$\Lambda = \ln K = \frac{\pi R}{\omega_e L}$$

II. $\omega_0 < \gamma$

$$I = A\,e^{\beta_1 t} + B\,e^{\beta_2 t}$$

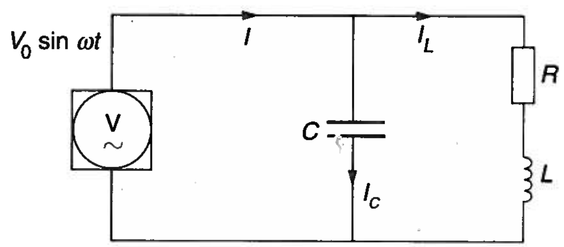
$$\beta_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

III. $\omega_0 = \gamma = -\beta$

$$I = e^{\beta t}(A + Bt)$$

A and *B* are constants

The Parallel Circuit



Resonant frequency

$$\omega_0' = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \omega_0\sqrt{1 - \frac{1}{Q_0^2}}$$

Impendance

$$Z = \frac{R + j\,\omega L}{1 - \omega^2 LC + j\omega RC} \qquad j = \sqrt{-1}$$

In particular, if $\omega = \omega_0'$

$$Z = R\,Q_0^2 = \frac{L}{RC}$$

Quality of the parallel circuit

$$Q_0' = \frac{L}{R}\,\omega_0' = Q_0\sqrt{1 - \frac{1}{Q_0^2}}$$

Band width (in Hz)

$$B = \frac{\omega_1 - \omega_2}{2\pi} \approx \frac{\omega_0}{2\pi Q_0}$$

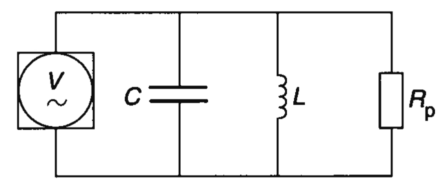
Formulae for $\omega = \omega_0'$, Q_0 ¡ 1

$$I_L \approx -j\,Q_0 I \qquad I_C \approx j\,Q_0 I$$

Approximate equivalent circuit when Q_0 ¡ 1

$$R_p = \frac{\omega_0^2 L^2}{R} Q_0^2 R$$

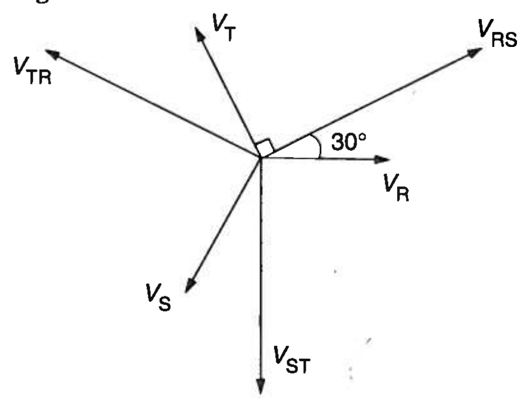
$$Z \approx \frac{R_p}{1 + j\,2\delta Q_0}$$



$$\delta = \frac{\omega - \omega_0}{\omega_0}$$ = relative frequency difference

3.14 Three-Phase Current

Voltage phasor diagram

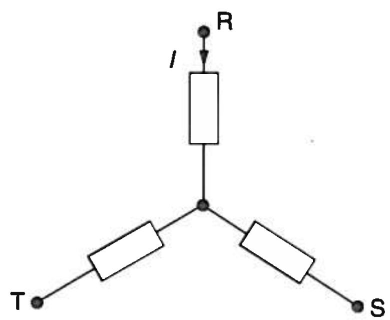


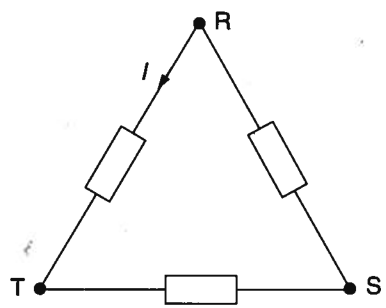
Line voltages V_{RS}, V_{ST}, and V_{TR} are denoted by V_h
Phase voltages V_R, V_S, and V_T are denoted by V_p

Relationship between line and phase voltages

$$V_h = \sqrt{3}\ V_p$$

Delta and Y connections, symmetrical load





$I = I_\ell$ | $I = \frac{1}{\sqrt{3}} I_\ell$

$P = \sqrt{3}\ V_h I_\ell \cos\varphi$ | $P = \sqrt{3}\ V_h I_\ell \cos\varphi$

I_ℓ = phase current (line current effective value). See also Sec. 3.11.

Definition of short-circuit power

$S_{SC} = \sqrt{3}\ V_h I_{SC}$ $\qquad$ I_{SC} = short-circuit current

Voltage drop in transmission lines

$$V_{1h} - V_{2h} = \sqrt{3}\ I_\ell\ (R\cos\varphi_2 + X\sin\varphi_2)$$

R and X = resistance and reactance of transmission lines (per phase)

$\cos\varphi_2$ = load power factor

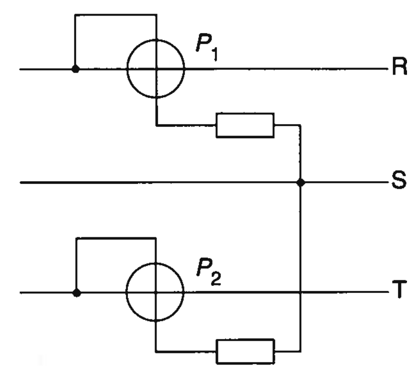
Power loss

$$P = 3\,R\,I_\ell^{\,2}$$

Measuring the power without a neutral by the two-wattmeter method

$$P = P_1 + P_2$$

$$\tan\varphi = \sqrt{3}\ \frac{P_2 - P_1}{P_2 + P_1}$$



3.15 Rotating Electric Machines

Mechanical power

$P = M\omega$ $\qquad$ M = torque

DC machines

Induced emf in rotor (armature)

$\varepsilon = k_1\,n\,\Phi_\delta$ $\qquad$ n = number of turns

Φ_δ = air-gap flux

$$V_a = \varepsilon + R_a I_a + V_c$$

R_a =resistance in armature winding
I_a =armature current
V_c =brush voltage drop

Torque

$$M = k_2 I_a \Phi_\delta$$

Air-gap flux

$$\Phi_\delta = \Phi_{\delta 0}\left(1 + \frac{k_s I_a - k_a |I_a|}{I_{an}}\right)$$

k_s = series winding constant (if no series winding, set $k_s = 0$)
k_a = armature reaction constant
I_{an} = rated current

Leakage flux

$$\Phi_{\delta 0} \approx k_3 I_m + \Phi_r$$

I_m = magnetization current
Φ_r = residual flux

The series motor

$$M \approx k_4 I_a^2$$

Generator operation

$I_a < 0 \qquad V_c < 0$

The Asynchronous Motor

Flux r.p.m. (synchronous r.p.m.)

$$n_1 = \frac{60 f_0}{p}$$

f_0 = applied voltage frequency
p = number of pole pairs

Lag

$$s = \frac{n_1 - n}{n_1}$$

n = r.p.m. of motor

$$\varepsilon_2 = s\, \varepsilon_{20}$$

ε_{20} = emf with stationary rotor

Rotor frequency

$$f_2 = s f_0$$

Motor torque

$$M \approx M_{max} \frac{2 s s_m}{s_m^2 + s^2}$$

$$s_m = \frac{R_2 + R_o}{X_{20}}$$

$$M_{max} = k_5 \frac{V_1^2}{X_{20}}$$

s_m = lag at maximum torque
R_2 = resistance of rotor winding (per phase)
R_o = outer resistance of rotor winding (per phase)
X_{20}= reactance of rotor winding (per phase) with stationary rotor
V_1 = applied voltage

In particular, when $s \ll s_m$

$$M = k_6 s V_1^2$$

Efficiency of motor

$$\eta = \frac{P_1 - P_p}{P_1} = \frac{P_2}{P_1}$$

P_1 = input electric power
P_2 = output mechanical power

$$P_p = P_0 + P_{b1} + P_{b2}$$

P_0 = leakage loss
P_{b1} = power loss in stator
$P_{b2} \approx s\,(P_1 - P_0 - P_{b1}) \approx$ power loss in rotor